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Pro-Cyclical Emissions, Real Externalities, and Optimal Monetary Policy*

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Abstract

We study optimal monetary policy in an analytically tractable New Keynesian DSGE-model with an emission externality. Empirically, emissions are strongly pro-cyclical and output in the flexible price equilibrium overreacts to productivity shocks, relative to the efficient allocation. At the same time, output under-reacts relative to the flexible price allocation due to sticky prices. Therefore, it is not optimal to simultaneously stabilize inflation and to close the natural output gap, even though this would be feasible. Real externalities affect the LQ-approximation to optimal monetary policy and we extend the analysis of Benigno and Woodford (2005) to inefficient flexible price equilibria. For central banks with a dual mandate, optimal monetary policy places a larger weight on output stabilization and targets a non-zero natural output gap, implying a higher optimal inflation volatility.

Keywords: Optimal Monetary Policy, Carbon Emissions, Output Gap, Central Bank Loss Function, Phillips Curve

JEL Classification: E31, E58, Q58

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1 Introduction

There is a broad consensus that the emission of greenhouse gases inflicts severe damages on the wider economy through their contribution to climate change. Economic theory suggests that Pigouvian emission taxes are the best instrument to address such an externality and it is becoming increasingly clear that central banks can play at most a supporting role in addressing externalities related to emissions. First, conventional monetary policy instruments, such as short term interest rates are naturally not well-suited to address long run issues such as climate change. Second, even the unconventional central bank toolkit provides very limited potential to induce a sectoral re-allocation away from fossil fuels.[1] However, it remains an unanswered question how to adapt monetary policy to a world characterized by climate change.

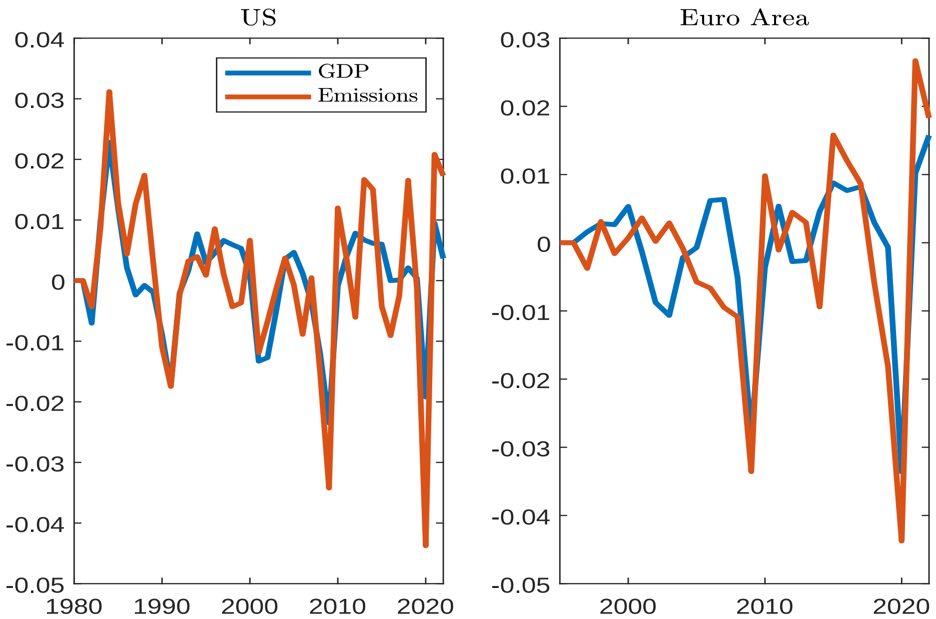
While climate change and socially harmful emissions affect the macroeconomy in many ways, we focus on a dimension that is by definition relevant for macroeconomic stabilization policies: emissions are highly pro-cyclical, both in the US and in the euro area, see Figure 1 for the case of carbon dioxide. Doda (2014) and Khan et al. (2019) provide additional evidence. Thus, a Pigouvian emission tax that addresses the emission externality in the long run but ignores the pro-cyclicality of emission damages does not implement the efficient allocation. Instead, output in the competitive equilibrium allocation under flexible prices overreacts to productivity shocks, relative to the efficient allocation.[2]

At business cycle frequencies, the relative overreaction of output in the flexible price equilibrium interacts non-trivially with nominal rigidities. Consider a positive shock to total factor productivity. A price setting friction a la Calvo (1983) prevents a fraction of firms from reducing prices, such that the economy expands by less than it would do under flexible prices. Absent emission externalities, the central bank aims at closing the gap between the sticky price and flexible price output. We refer to this gap as the *natural* output gap. With pro-cyclical emissions, closing the natural output gap does not implement the efficient allocation. From a normative point of view, we refer to the difference between the output reaction under sticky prices minus the output reaction in the efficient allocation as the *efficient* output gap. Price stickiness attenuates the overreaction of the competitive equilibrium vis-a-vis the efficient allocation. Therefore, the efficient output gap is ambiguously affected by the emission externality: while it is gener-

[1] We refer to Giovanardi et al. (2023) for an assessment of preferential collateral haircuts for green bonds and to Ferrari and Nispi Landi (2023) for green QE.

[2] Benmir, Jaccard, and Vermandel (2024) argue that household stochastic discount factors that are consistent with macro-finance moments yield a strongly pro-cyclical social cost of carbon. This strengthens our argument that relies on the overreaction of economic activity to productivity shocks, relative to the planner solution.

Figure 1: Carbon Emissions and GDP over Time



Notes: Data at annual frequency, detrended using a one-sided HP-filter with smoothing parameter 6.25. The full-sample correlations are 0.78 for the US and 0.77 for the Euro Area.

ally larger than the natural output gap it can be positive or negative. This implies that, in contrast to the standard New Keynesian model, the central bank is unable to perfectly stabilize both inflation and the efficient output gap. Equivalently, the central bank could perfectly stabilize inflation and natural output gap, but this is not optimal. Divine coincidence as defined by Blanchard and Gali (2007) is broken.[3]

We incorporate this insight into an otherwise standard New Keynesian model with nominal rigidities (Calvo 1983) and socially harmful emissions, similar to

[3]Breaking divine coincidence in the presence of productivity shocks requires frictions that go beyond nominal rigidities. For example, Faia (2009) shows that search frictions on the labor market render the flexible price allocation infeasible. In contrast, the flexible price allocation is implementable in our framework, but it is not optimal to do so. Adao, Correia, and Teles (2003) demonstrate that in an economy with cash-in-advance constraints, it is not optimal to fully stabilize prices and output gaps, which is conceptually similar to our results. Sims, Wu, and Zhang (2023) discuss the role of financial shocks as inflation shifters in the New Keynesian Phillips curve, which also break divine coincidence.

the four-equation New Keynesian model analyzed by Sahuc, Smets, and Vermandel (2024). As far as the over-reaction of output in the flexible price equilibrium is concerned, we show that the four-equation model can be reasonably approximated by a three-equation model where emission damages depend on the flow of emissions. The reason for this approximate equivalence is that output slightly over-reacts for many periods with persistent emissions, while it strongly but briefly over-reacts in the three-equation model. The planner takes this time pattern of the over-reaction into account, such that the discrepancy between efficient and natural output reaction is approximately independent of the degree of emission persistence.

As our main contribution, we present an analytical characterization of optimal monetary policy in the three-equation model along the lines of Clarida, Galí, and Gertler (1999) and Woodford (2011). Our analysis is applicable for central banks with a dual mandate. Conceptually, optimal monetary policy under cyclical emissions is a second best solution to a utilitarian welfare-maximization problem. With a time-invariant tax, optimal monetary policy addresses two dynamic frictions with only one instrument - the nominal interest rate - and will not be able to offset both inefficiencies at once. If appropriate cyclical adjustments to emission taxes were in place, optimal monetary policy could be conducted as usual.[4] The presence of a second dynamic inefficiency affects the central bank's objective function, which is derived from first principles.

It is the derivation of the central bank loss function for an inefficient competitive equilibrium where our key methodological innovation lies. As customary in the literature, the loss function builds on a second order approximation to the household utility function and uses equilibrium conditions to express this in terms of output gap and inflation. It turns out to be essential to include an additional condition that takes the effect of economic activity on emission damages into account. Without this additional condition, the central bank would simply take emission damages as given and fail to internalize the negative emission externality. Put differently, the resulting loss function would inherit the market failure from the flexible price equilibrium and prescribe to close inflation and natural output gap in all states.

The additional expression, which ensures that the loss function internalizes the emission externality, introduces linear terms related to the *level* of emission damages into the loss function. In order to cleanly separate business cycle stabilization objectives from expressions related to steady state inefficiencies, we eliminate those

[4]It appears rather implausible from an institutional background that central banks can directly address pro-cyclical emissions, for example by purchasing and selling emission permits. Arguably, climate policy is usually conducted over long run horizons, while overreactions of output to business cycle fluctuations typically belong to the domain of macroeconomic stabilization policies.

terms by a second order approximation of the relationship between economic activity and emission damages. This step is in the spirit of Benigno and Woodford (2005), who use a second order approximation of the Phillips curve for monopolistic distortions in steady state.[5] The resulting loss function features a non-zero target level for the natural output gap that reflects the over-reaction of output in competitive equilibrium relative to the efficient allocation. The target level is negative for a positive TFP shock and increasing in absolute terms in the severity of the emission externality. In addition, optimal monetary policy places a higher weight on output stabilization if the externality is more severe, but is independent of the steady state distortion as in Benigno and Woodford (2005). This result resonates with Clarida, Galí, and Gertler (1999), who show that the weight on inflation stabilization is higher if the output is *below* its efficient level due to monopolistic distortions. Here the opposite intuition applies, since output is above the efficient level.

Equipped with the central bank loss function, the LQ-approach to optimal monetary under discretion then simply minimizes the loss function subject to the New Keynesian Phillips curve. Here, the central bank takes into account that the Phillips curve steepens in the presence of socially harmful emissions: since emission damages exert a general dampening effect on economic activity, output responds by less for a given change in the price level. The solution to the LQ-problem shows that the optimal natural output gap is a linear combination between its target level and zero, i.e. the flexible price allocation. The (optimal) natural output gap is closer to its target level if the emission externality is large, both through its effect on the weight and the slope of the Phillips curve. To dampen the output reaction, the central bank cuts interest rates by less in absolute terms in response to a positive TFP shock than it would to absent emission damages. Consequently, the central bank allows for some dis-inflation after a positive productivity shock. Optimal inflation volatility is larger in the presence of (pro-cyclical) emission externalities.

Quantitatively, we find that, in response to a positive one standard deviation TFP shock, the optimal interest rate cut is around 6 basis points smaller than the natural rate decline. We show numerically that the adjustment term in the policy rate is also remarkably similar in the four-equation model, even though the natural rates of interest differ quite substantially. This shows again that our analysis of *optimal* monetary policy is orthogonal to an analysis of the effects of climate change and socially harmful emissions on the natural interest rate r^*. Furthermore, the welfare gain of optimal monetary policy is smaller than in a counterfactual economy without emissions, as the central bank addresses an additional friction with its only

[5] We also approximate the Phillips curve up to second order in order to allow for large steady state discrepancies between efficient allocation and flexible price equilibrium, for example due to the absence of carbon taxes.

policy instrument, the nominal interest rate.

By providing a simple analytical framework, our analysis contributes to the understanding welfare-relevant output gaps, which are not only relevant for monetary policy frameworks in all jurisdictions that provide their central bank with a dual mandate, but for all macroeconomic policies that take output gaps into account. Whenever macroeconomic stabilization policies depend on output gaps, one has to bear in mind that those output gaps need not be efficient from a welfare perspective.[6] In spirit of the analysis by Blanchard and Gali (2007), we have shown how optimal monetary policy is affected by externalities originating in the real sector, which do not have a direct effect on nominal rigidities. While the flexible price allocation can be implemented in our model, it is not optimal to do so.

Related Literature Our paper draws from the E-DSGE literature that studies the macroeconomic effects of climate policies at business cycle frequencies, starting with the contribution by Heutel (2012). Consequently, this model class is suitable to study the relationship between environmental and monetary policies, see Annicchiarico et al. (2021) for a survey. Related to monetary policy, Annicchiarico and Di Dio (2015) and Annicchiarico and Di Dio (2017) study the interplay of nominal rigidities and different environmental policies, taking into account costly emission abatement at the firm level. Faria, McAdam, and Viscolani (2022) discuss the neutrality of monetary policy under different monetary frictions, such as cash-in-advance or money-in-the-utility function.

We contribute to a growing literature studying how monetary policy optimally adapts to climate change. McKibbin et al. (2020) provide an overview about potential interactions between climate policy and monetary policy. For a general discussion of these interactions, we also refer to Hansen (2021). Using the New-Keynesian framework, Muller (2023) proposes a concept for a real interest rate that takes time-varying pollution intensities into account. By tracking such a refined "green interest rate", monetary policy intertemporally re-allocates consumption from periods with high pollution intensity to periods with a low pollution intensity. Nakov and Thomas (2023) show that climate change, i.e. the long run consequences of emissions, only has a limited impact on the conduct of monetary policy. Economides and Xepapadeas (2025) study monetary policy numerically in a larger E-DSGE model, where positive TFP shocks have negative side effects through elevated damages from climate change. Our paper differs from these pa-

[6]On a conceptual level, our analysis also relates to the literature of optimal monetary policy in the presence of hysteresis effects. If such effects are present, it is not optimal to close the natural gap. In sharp contrast to a setting with emission externalities, however, optimal monetary policy is more expansionary in response to a positive TFP shock than in the standard New Keynesian model, see Cerra, Fatás, and Saxena (2023) and the references therein.

pers by focusing on optimal monetary policy and the characterization of central bank loss functions.

A series of papers studies monetary policy when inflation is partially driven by rising energy prices. In a New Keynesian model with an energy sector, Olovsson and Vestin (2023) show that targeting core inflation is welfare-optimal. The literature also recognizes that monetary policy might be affected by potentially inflationary effects of carbon taxation more generally. Konradt and Weder di Mauro (2023) and Hensel, Mangiante, and Moretti (2024) provide empirical evidence. Ferrari and Nispi Landi (2022), Del Negro, Di Giovanni, and Dogra (2023) and Airaudo, Pappa, and Seoane (2024) study this channel through the lenses of small- to medium-scale New Keynesian models, while Sahuc, Smets, and Vermandel (2024) estimate a New Keynesian E-DSGE model to assess the macroeconomic relevance of "climateflation" and "greenflation".

Outline Our paper is structured as follows. Section 2 presents a four-equation New Keynesian model, augmented by a law of motion for socially harmful emissions. Section 3 demonstrates that the natural and welfare-relevant output gaps can be reasonably approximated in a reduced three-equation version of the model and then characterizes the competitive equilibrium, holding monetary policy constant. In Section 4, we derive the central bank loss function and study optimal monetary policy in closed form for the case i.i.d. productivity shocks. Section 5 provides a quantitative exploration of optimal monetary policy for the general case of persistent shocks and slowly accumulating emissions. Section 6 concludes.

2 Model

We present the basic monetary policy trade-off in an otherwise standard New Keynesian model, augmented by socially harmful emissions. There is a representative household, monopolistically competitive firms, a fiscal authority, and the central bank. Emissions negatively affect the productivity of final good producers through a damage function.[7]

Households The representative household saves using nominal deposits S_t that pay the one-period gross interest rate r_t^s, consumes the final consumption good c_t, and supplies labor n_t at the nominal wage W_t. The household also owns firms and receives their profits d_t^{firms}, expressed in real terms. The maximization problem

[7]Analytically similar results can be obtained by assuming that emissions exert a utility loss on households. In this case, the competitive equilibrium under flexible prices also overreacts to TFP shocks.

is given by

$$\max_{\{c_t,n_t,S_t\}_{t=0}^{\infty}} \quad \mathbb{E}_0 \left[\sum_{t=0}^{\infty} \beta^t \left(\frac{c_t^{1-\sigma}-1}{1-\sigma} - \frac{n_t^{1+\varphi}}{1+\varphi} \right) \right]$$
$$\text{s.t.} \quad P_t c_t + S_t = W_t n_t + r_{t-1}^s S_{t-1} + P_t d_t^{firms} \ .$$

The parameters σ and φ determine the inverse of, respectively, the intertemporal elasticity of substitution and the elasticity of labor supply. Solving this maximization problem yields a standard Euler equation and an intra-temporal labor supply condition

$$c_t^{-\sigma} = \beta r_t^s \mathbb{E}_t \left[\frac{c_{t+1}^{-\sigma}}{\pi_{t+1}} \right] \ , \tag{1}$$

$$n_t^{\varphi} = w_t c_t^{-\sigma} \ . \tag{2}$$

Here, P_t is the price level, $w_t \equiv \frac{W_t}{P_t}$ is the real wage, and $\pi_t \equiv \frac{P_t}{P_{t-1}}$ denotes gross inflation.

Firms: Technology There is a mass-one continuum of monopolistic firms, indexed by i. Firm i hires labor $n_t(i)$ to produce the intermediate good $y_t(i)$ with the following technology:

$$y_t(i) = \Lambda_t A_t n_t(i) \ . \tag{3}$$

During the production process, firms emit one unit of socially harmful emissions per unit of output. The stock of atmospheric carbon dioxide E_t evolves according to

$$E_t = y_t + (1-\delta) E_{t-1} \ , \tag{4}$$

where $\delta > 0$ is the depreciation rate of atmospheric carbon dioxide. Following the literature, emissions endogenously reduce productivity

$$\Lambda_t = \exp\left\{-\Gamma E_t\right\} \ , \tag{5}$$

where the parameter Γ governs the severity of damages associated with emissions. Importantly, damages are an externality, because they depend on aggregate economic activity y_t, which individual firms take as given. Our analysis abstracts from technological change or abatement effort at the firm level.

Production is taxed at the potentially time-varying rate τ_t^c. As we shall see, optimal emission taxes are pro-cyclical in this setup. This resembles the result of Golosov et al. (2014), who show that optimal emission taxes are proportional to

GDP growth. Crucially, whenever emission taxes do not respond appropriately to the business cycle, the model features a dynamic inefficiency that affects the optimal conduct of monetary policy.[8]

Before introducing nominal rigidities, some remarks on emission damages are in order. While our quantitative application focuses on the negative effects of carbon emissions through climate change, our analysis is also applicable to emission damages beyond climate change. The environmental economics literature typically views climate change as only a subset of the overall adverse effects that the emissions exert on the wider economy. Other adverse effects include negative health consequences through air quality losses, decreased timber and agriculture yields, depreciation of materials, and reductions of recreation services. For details, we refer to Muller, Mendelsohn, and Nordhaus (2011) and the references therein. In contrast to climate change, these negative effects materialize very quickly in response to a cyclical increase in economic activity, but also depreciate faster. More generally, our analysis is also applicable to the pro-cyclical depletion of renewable resources and other real externalities, such as congestion, that are positively associated with economic activity.

These alternative interpretations will of course have different quantitative implications for the optimal conduct of monetary policy. Specifically, the elasticity of emission damages with respect to current output, the depreciation rate of pollutants, and the recovery rate of renewable resources has a large effect on the *natural rate* in these different economies. Notably, emission damages drives a very similar wedge between efficient and natural level of output, irrespective of their persistence and their effect on *current* damages. This is because the planner takes the *present value* of all future damages into account when deciding on the socially optimal output expansion. If pollution damages depend on a slowly depreciating stock of emissions, output expansions have only little effect on current damages, but long-lasting effect on future damages. Conversely, if damages depend on the flow, there is large negative effect on impact but now effect on household welfare in future periods. In the next section, we show that the discrepancy between efficient and natural output expansion are quantitatively small.

Nominal Rigidities The rest of the supply side coincides with the standard New Keynesian model: monopolistic producers are not perfectly able to adjust their prices due to nominal rigidities, modeled as in Calvo (1983), with θ being the fraction of firms that is not allowed to change prices. The optimal price for a

[8]Note that our analysis is based on a stationary model. If climate policy is instead modeled in terms of a transition towards higher taxes, optimal taxes should still be above (below) trend during a boom (recession). As long as the carbon taxes do not deviate from their trend in response to business cycle fluctuations, the dynamic inefficiency arises.

firm that is able to adjust prices is given by

$$p_t^* = \frac{1}{1-\tau_t^c}\frac{\epsilon}{\epsilon-1}\frac{\xi_{1,t}}{\xi_{2,t}} ,$$

where

$$\xi_{1,t} = mc_t\, y_t + \beta\, \theta\, \mathbb{E}_t \left[\frac{c_{t+1}^{-\sigma}}{c_t^{-\sigma}} \pi_{t+1}^{\epsilon} \xi_{1,t+1} \right] \quad \text{and} \quad \xi_{2,t} = y_t + \beta\, \theta\, \mathbb{E}_t \left[\frac{c_{t+1}^{-\sigma}}{c_t^{-\sigma}} \pi_{t+1}^{\epsilon-1} \xi_{2,t+1} \right] .$$

Carbon taxes enter the model by increasing firms marginal cost, which implies that the optimal price of an adjusting firm p_t^* increases in the carbon tax τ_t^c. The nominal friction implies that monopolistic producers face time-varying real marginal costs, thus generating a relationship between inflation and real economic activity summarized in the New-Keynesian Phillips Curve.[9] Exogenous total factor productivity A_t follows an AR(1) process in logs:

$$\log(A_t) = \rho_A \log(A_{t-1}) + \sigma_A \epsilon_t \ , \quad \text{where } \epsilon_t \sim N(0,1) \ .$$

The full list of equilibrium conditions is summarized in Appendix A.1, equations (A.8)-(A.18).

Parameterization While all of our main results are obtained analytically, we illustrate their quantitative relevance under a standard quarterly parameterization following the New Keynesian DSGE literature. Households' risk aversion and discount factor are set to $\sigma = 1$ and $\beta = 0.995$. This discount factor implies an annual real rate of 2%. Furthermore, we set $\varphi = 4/3$, following Chetty et al. (2011).

Regarding the emission externality, we use a narrow interpretation as carbon emissions and target recent estimates by Bilal and Kaenzig (2024), who identify a long run productivity loss of 31% in an economy without climate policy. We perform a change of variables and define the steady state output-adjusted damage parameter $\gamma \equiv \frac{\Gamma}{E}$. Crucially, E corresponds to steady state output in the economy without carbon taxes. Setting $\gamma = 0.37$ yields a long run value of $\Lambda = 0.69$ in the model. Under this parameterization, the optimal long run tax in the model is given by $\tau_c = \frac{0.37}{1+0.37} \approx 0.27$. Following Gibson and Heutel (2023), we set the quarterly depreciation rate to $\delta = 0.0035$, which implies a 50 year half-life for atmospheric carbon dioxide.

The demand elasticity for final good varieties is fixed at $\epsilon = 6$, implying a 20% markup. As a benchmark, we set the Calvo parameter to $\theta = 0.8$, corresponding

[9]Similar results can be obtained by imposing price adjustment costs instead of staggered pricing.

Table 1: Parameterization

Parameter	Value	Source/Target
Households		
Household discount factor β	0.995	Standard
Consumption CRRA σ	1	Log-utility
Labor supply curvature φ	4/3	Chetty et al. (2011)
Technology		
Calvo parameter θ	0.8	Price Duration 5 Quarters
Emission damage γ	0.37	Bilal and Kaenzig (2024)
Emission decay δ	0.0035	Gibson and Heutel (2023)
Shocks		
Persistence TFP ρ_A	0.9	Standard
TFP shock st. dev. σ_A	0.01	Standard

to an expected price duration of five quarters. In a comparative statics exercise, we decrease this parameter to $\theta = 0.6$, implying an expected price duration of 2.5 quarters. The parameters governing exogenous TFP are set to $\rho_A = 0.9$ and $\sigma_A = 0.01$. The parameterization is summarized in Table 1.

3 Exogenous Monetary Policy

Having described the model, we first discuss the positive properties of augmenting an otherwise standard New Keynesian model with socially harmful emissions. To do so, we keep the monetary policy reaction function constant. For the remainder of this paper, we assume that the fiscal authority sets a constant labor subsidy, $\tau^n = \frac{1}{\epsilon}$, which implies $(1 - \tau^n)\mu = 1$ and eliminates the steady state distortion generated by monopolistic competition. Since emission externalities are the only inefficiency in the flexible price equilibrium, we can cleanly distinguish between natural and efficient (welfare-relevant) output gap. We then demonstrate that the dynamics of both output gaps can be accurately approximated by assuming that damages depend on current output rather than the stock of carbon dioxide emissions. This will allow us to characterize optimal monetary policy and the central bank loss function in closed form in the next section.

3.1 Efficient and Natural Output Gap

We begin by characterizing the efficient output y_t^e and the natural output y_t^n, defined as the output consistent with perfectly flexible prices, as functions of total factor productivity A_t.

Proposition 1. The natural y_t^n and efficient y_t^e output levels can be written as a function of the only state variable A_t:

$$(y_t^n)^{\sigma+\varphi} = (1-\tau_t^c)(A_t\Lambda_t)^{1+\varphi} \ . \tag{6}$$

$$(y_t^e)^{-\sigma} - \frac{(y_t^e)^{\varphi}}{(A_t\Lambda_t)^{1+\varphi}}\big(1+\gamma\frac{y_t^e}{E}\big) = \beta(1-\delta)\big((y_{t+1}^e)^{-\sigma} - \frac{(y_{t+1}^e)^{\varphi}}{(A_{t+1}\Lambda_{t+1})^{1+\varphi}}\big) \ . \tag{7}$$

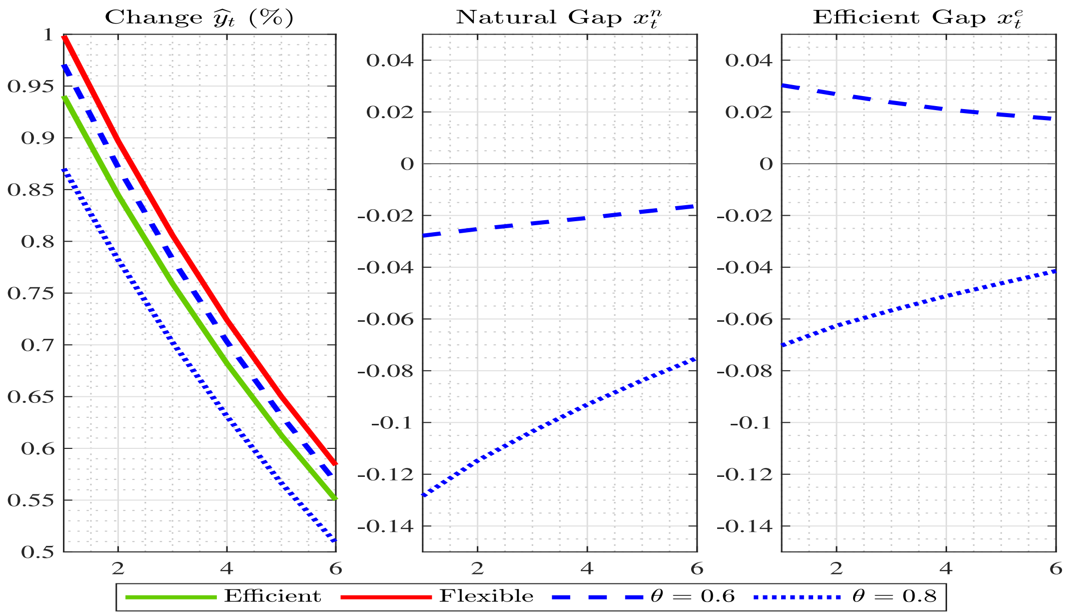
Proof: see Appendix A.1.

These output responses serve as reference in the definition of the efficient $x_t^e \equiv \widehat{y}_t - \widehat{y}_t^e$ and the natural output gap $x_t^n \equiv \widehat{y}_t - \widehat{y}_t^n$, respectively. Here $\widehat{y}_t$ is the output deviation from its steady state level for any given degree of nominal rigidities and any given monetary policy. Without externalities ($\gamma = 0$), the flexible price competitive equilibrium coincides with the efficient allocation, such that both output gaps coincide $x_t^e = x_t^n$.

With externalities ($\gamma > 0$), an overreaction of the flexible price economy in response to a positive TFP shock implies a positive efficient output gap, illustrated in Figure 2. The solid red line in the left panel corresponds to the output expansion in the flexible price equilibrium, which exceeds the efficient output expansion, indicated by the solid green line. The output expansion with sticky prices is smaller than in the flexible price case, as shown by the dashed blue line for a Calvo parameter of $\theta = 0.6$ and the dotted blue line for $\theta = 0.8$. Consequently, as the middle panel of Figure 3 shows, the natural output gap in response to a positive TFP shock is negative.

The right panel shows that it depends on the relative strength of nominal rigidities and the emission externality, whether the competitive equilibrium still overreacts relative to the efficient allocation. For a large θ, nominal rigidities dominate and the efficient output gap is also negative, albeit smaller. For smaller values of θ, the overreaction of output with respect to the efficient allocation dominates and the efficient output gap turns positive. In our parameterization, this happens for a Calvo parameter between 0.6 and 0.8, i.e. for relevant parts of the parameter space. It will turn out that the interaction of these two dynamic inefficiencies, nominal rigidities and emission externalities, is non-trivial and bears direct implications for the conduct of monetary policy.

Figure 2: IRF to TFP-Shocks: Baseline Four Equation Model



Notes: Impulse responses to a positive one standard deviation shock to TFP. Output change $\widehat{y}_t$ is expressed in relative deviations from its steady state value.

3.2 Eliminating One Equation

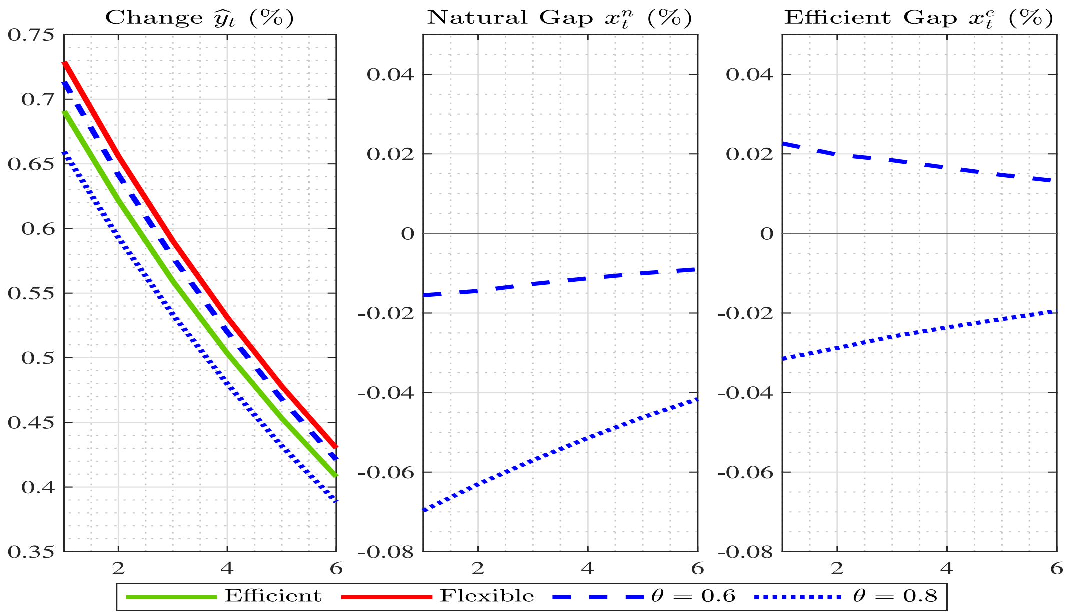
Similar to the setup studied in Sahuc, Smets, and Vermandel (2024), the characterization of this model is complicated by the presence of the law of motion for emissions. While Sahuc, Smets, and Vermandel (2024) estimate such a four-equation New Keynesian model, the lower panel of Figure 3 reveals that eliminating the law of motion for emissions by setting $\delta_E = 1$ delivers a reasonably good approximation of the efficient and natural output gap, respectively, which are the main focus of our optimal monetary policy analysis. Notably, the output reaction $\widehat{y}_t$ is smaller in the model with full depreciation, as damages Λ_t is more responsive to current economic activity. However, the response of actual and efficient output relative to the flexible price output reaction are remarkably similar. The reason is that the social planner takes the *present value* of future damages into account when computing the optimal output expansion.

Lemma 1. If $\delta_E = 1$, the efficient output level y_t^e reduces to:

$$(y_t^e)^{\sigma+\varphi} = \frac{(A_t\Lambda_t)^{1+\varphi}}{1+\gamma\frac{y_t^e}{y}}\,, \tag{8}$$

which immediately obtains from simplifying (7). Combining (8) with (6), the ratio

Figure 3: IRF to TFP-Shocks: Simplified Three Equation Model



Notes: Impulse responses to a positive one standard deviation shock to TFP. Output change $\widehat{y}_t$ is expressed in relative deviations from its steady state value.

of natural and efficient output simplifies to

$$\left(\frac{y_t^n}{y_t^e}\right)^{\sigma+\varphi} = \left(1+\gamma\frac{y_t^e}{y}\right)(1-\tau_t^c)\,. \tag{9}$$

Lemma 1 is a straightforward simplification of Proposition 1. It follows from equation (9) that a time-varying emission tax $\tau_t^c = \frac{\gamma\frac{y_t^e}{y}}{1+\gamma\frac{y_t^e}{y}}$ implements the efficient allocation. In this case, the RHS of (9) collapses to one and natural and efficient output coincide.

To facilitate an analysis of the optimal monetary policy problem, we then express their responses $\widehat{y}_t^n$ and $\widehat{y}_t^e$ to a technology shock a_t in deviations from steady state.

Lemma 2. Their log-deviations around the deterministic steady state are given by:

$$\widehat{y}_t^n = \frac{1+\varphi}{\zeta}a_t - \frac{1}{\zeta}\frac{\tau^c}{1-\tau^c}\widehat{\tau}_t^c \tag{10}$$

$$\widehat{y}_t^e = \frac{1+\varphi}{\widetilde{\zeta}+\widetilde{\gamma}}a_t\,, \tag{11}$$

where

$$\zeta \equiv \varphi + \gamma(1+\varphi) + \sigma, \quad \widetilde{\zeta} \equiv \varphi + \gamma \frac{y^e}{y}(1+\varphi) + \sigma \quad \text{and} \quad \widetilde{\gamma} \equiv \frac{\gamma}{1+\gamma} . \tag{12}$$

Proof: see Appendix A.2.

Lemma 2 demonstrates that, all else equal, positive TFP shocks increase output while a positive carbon tax shock is recessionary. Furthermore, absent emission taxes ($\tau_t^c = 0$), the natural level of output generally exceeds its efficient level. Notably, the recessionary effect of the carbon tax precisely addresses over-production in the competitive equilibrium.[10] Combining (8) and (6), However, even with a carbon tax implementing the efficient steady state output, emissions still generate a dynamic inefficiency. Specifically, with $\tau^c = \widetilde{\gamma}$ and $\widehat{\tau}_t^c = 0$, output in the competitive equilibrium $\widehat{y}_t^n$ overreacts to technology shocks relative to the efficient allocation $\widehat{y}_t^e$. Indeed in this case $\widetilde{\zeta} = \zeta$, implying $\widehat{y}_t^e = \frac{1+\varphi}{\zeta+\widetilde{\gamma}} < \frac{1+\varphi}{\zeta} = \widehat{y}_t^n$. Since this dynamic inefficiency is the key element of our analysis, we focus on the empirically plausible case of constant carbon taxes ($\widehat{\tau}^c = 0$) in the following characterization of monetary policy.[11]

3.3 Characterization of the Three-Equation Model

As a next step, we flesh out the interactions between nominal rigidities and procyclical emissions for an exogenously given nominal interest rate r_t^s in the three-equation model. Proposition 2 shows that its equilibrium is characterized by a dynamic IS curve and the New Keynesian Phillips curve.

Proposition 2. The equilibrium conditions for the economy with nominal rigidities simplify to the following two linear conditions in terms of log-deviations from the steady-state:

$$x_t^n = \mathbb{E}_t[x_{t+1}^n] - \frac{r_t^s - \mathbb{E}_t[\pi_{t+1}]}{\sigma} + \underbrace{\frac{1}{\zeta}\mathbb{E}_t\left[(1+\varphi)(a_{t+1}-a_t) - \frac{\tau^c(\widehat{\tau}_{t+1}^c - \widehat{\tau}_t^c)}{1-\tau^c}\right]}_{=r_t^n/\sigma} , \tag{13}$$

$$\pi_t = \zeta\kappa x_t^n + \beta\mathbb{E}_t[\pi_{t+1}] + \beta(1-\theta)\frac{\tau^c}{1-\tau^c}(\widehat{\tau}_t^c - \mathbb{E}[\widehat{\tau}_{t+1}^c]) . \tag{14}$$

[10] The same outcome can also be achieved by a cap-and-trade policy. Since we assume that one unit of production entails one unit of emissions, a policy that issues y_t^e certificates each period implements the efficient allocation. If the market for emission certificates is frictionless, the permit price corresponds to the optimal carbon tax.

[11] In practice, carbon taxes or emission trading systems follow deterministic trends. As customary in the (New Keynesian) business cycle literature, the data counterparts of our model variables are trend deviations. Therefore, our results are relevant whenever climate policy does not respond to trend deviations, just as the concept of pro-cyclical emissions in Figure 1 is based on trend deviations of GDP and emissions.

Proof: see Appendix A.3.

Equation (13) is a dynamic IS curve: the (natural) output gap x_t^n positively depends on the expected output gap next period and negatively depends on the real interest rate gap $\frac{r_t^s - \mathbb{E}_t[\pi_{t+1}] - r_t^n}{\sigma}$, defined as the real interest rate, $r_t^s - \mathbb{E}_t[\pi_{t+1}]$, minus the natural real interest rate, r_t^n. The natural interest rate is the real interest rate consistent with the natural level of output, which is in turn defined as the level of output consistent with flexible prices.[12]

The New Keynesian Phillips curve is given by equation (14). As usual, its slope depends on nominal rigidities through the auxiliary parameter $\kappa \equiv \frac{(1-\theta\beta)(1-\theta)}{\theta}$. In the presence of emission externalities, the slope is also affected by the auxiliary parameter ζ as defined in equation (12). Since ζ is positively related to the severity of the emission externality, we can easily asses how the emission externality affects the Phillips curve (14). Since the Phillips curve relates the natural output gap to inflation, it is helpful to discuss the effect of emissions on output gap and inflation separately. On the one hand, the inflation response to a TFP shock is determined by the share of firms that can reduce their price, which does not depend on the emission externality. On the other hand, the emission externality dampens the effects of a TFP shock on the output gap. This implies that, for a given output gap, inflation responds more strongly to a TFP shock if $\gamma > 0$. Consequently, pro-cyclical emissions steepen the Phillips curve.

Note that these inflationary pressures, sometimes referred to as *climateflation*, do not imply that the emission externality is inflationary in equilibrium, as firms take expected inflation in future periods into account when setting their prices today. It is, therefore, necessary to iterate forward the Phillips curve when characterizing the equilibrium effect of TFP shocks. Since expected inflation depends on the central bank's reaction to shocks, we close the model with a Taylor-type rule for the nominal interest rate:

$$r_t^s = \overline{r}^s \cdot \pi_t^{\phi} \,, \tag{15}$$

where ϕ governs the response of the short term nominal interest rates to inflation and $\overline{r}^s = 1/\beta$ is the steady state real interest rate.

Proposition 3 characterizes how pro-cyclical emissions affect output and inflation in the competitive equilibrium for the constant central bank reaction function (15).

[12] As a by-product of our analysis, equations (13) and (14) characterize the effects of carbon taxes on inflation and output. Specifically, transitory carbon tax shocks, i.e. $\widehat{\tau}_t^c > 0$ and $\widehat{\tau}_{t+1}^c = 0$, are both inflationary and recessionary. This result is consistent with empirical findings in Kaenzig (2023) and is related to the negative effect of carbon taxes on marginal costs, which implies that, on aggregate, firms reduce their production and increase their prices.

Proposition 3. Under time-invariant emission taxes, the policy functions for output gap and inflation read

$$x_t^n = \frac{\sigma}{\zeta} \cdot \frac{(1+\varphi)(1-\beta\rho_a)}{\sigma(1-\beta\rho_a)(1-\rho_a)+\zeta\kappa(\phi-\rho_a)} \cdot (\rho_a - 1)a_t \equiv \Theta_{xa}a_t \ , \tag{16}$$

$$x_t^e = \widetilde{\gamma}\frac{1+\varphi}{\zeta(\zeta+\widetilde{\gamma})} + \Theta_{xa}a_t \ , \tag{17}$$

$$\pi_t = \sigma\kappa \cdot \frac{1+\varphi}{\sigma(1-\beta\rho_a)(1-\rho_a)+\zeta\kappa(\phi-\rho_a)} \cdot (\rho_a - 1)a_t \equiv \Theta_{\pi a}a_t \ . \tag{18}$$

Moreover, the variances of output gap and inflation are given by:

$$Var[x_t^n] = \Theta_{xa}^2\sigma_A^2, \quad Var[\pi_t] = \Theta_{\pi a}^2\sigma_A^2 \ .$$

Proof: By the method of undetermined coefficients. Guess a linear policy function for $x_t^n = \Theta_{xa}a_t$ and $\pi_t = \Theta_{\pi a}a_t$, and impose equilibrium consistency in equation (13), equation (14), and equation (15), together with $\mathbb{E}_t[a_{t+1}] = \rho_a a_t$ and $\tau_t^c = 0$ to get:

$$\Theta_{xa}a_t = \Theta_{xa}\rho_a a_t - \frac{\phi\Theta_{\pi a}a_t - \Theta_{\pi a}\rho_a a_t}{\sigma} + \frac{1}{\zeta}\Bigg[(1+\varphi)(\rho_a a_t - a_t)\Bigg] \ ,$$

$$\Theta_{\pi a}a_t = \zeta\kappa\Theta_{xa}a_t + \beta\Theta_{\pi a}\rho_a a_t \ .$$

For the guess to be correct, the last two equations have to hold for each $a_t \in \mathcal{R}$. Hence, imposing $a_t = 1$ and solving the system of the two equations into the two unknowns, $\Theta_{\pi a}$ and Θ_{xa} yields the coefficients of the policy functions:

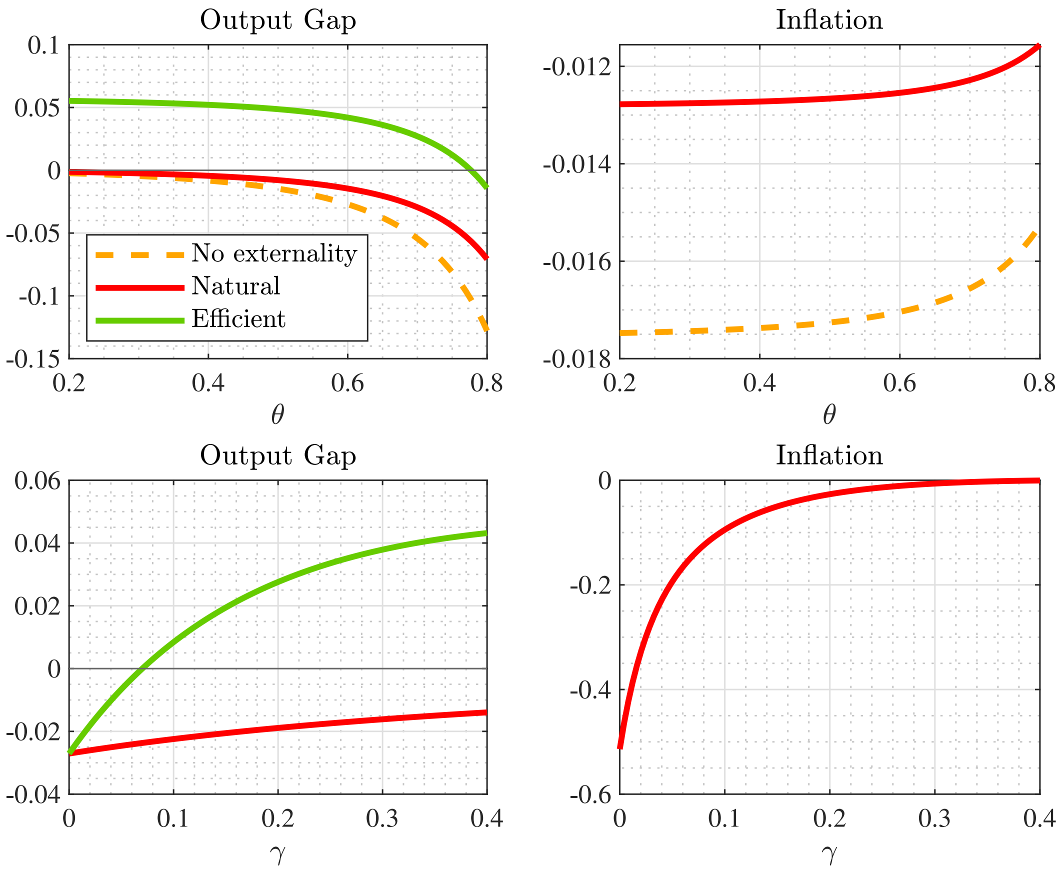
$$\Theta_{xa} = \frac{\sigma}{\zeta} \cdot \frac{(1+\varphi)(1-\beta\rho_a)}{\sigma(1-\beta\rho_a)(1-\rho_a)+\zeta\kappa(\phi-\rho_a)} \cdot (\rho_a - 1) \ , \tag{19}$$

$$\Theta_{\pi a} = \sigma\kappa \cdot \frac{1+\varphi}{\sigma(1-\beta\rho_a)(1-\rho_a)+\zeta\kappa(\phi-\rho_a)} \cdot (\rho_a - 1) \ . \tag{20}$$

□

Figure 4 illustrates Proposition 3 graphically. The first row shows the impact response of inflation and output gap to a positive technology shock as a function of θ. The standard case without emission externalities is indicated by the dashed yellow line. The output gap (inflation) is larger (smaller) in absolute terms as θ increases, i.e. as prices become more rigid. For the case with emission externalities we differentiate between the natural output gap x_t^n (red line) and the efficient output gap x_t^e (green line), which coincide for the standard model. Due to the dampening effect of emissions on TFP, the inflation and natural output gap are

Figure 4: Policy functions and variances as functions of θ and γ



less responsive to productivity shocks in the model with emissions but always negative. The response of the efficient output gap (17) is similar to the natural output gap, but shifted by a positive intercept term. The efficient output gap turns positive for $\theta < 0.78$, consistent with Figure 3.

In the second row, we plot the responses of both output gaps and inflation for different values for the emission externality. Differentiating the coefficients of the policy functions (19) and (20) with respect to γ, we immediately obtain $\frac{\partial \Theta_{xa}}{\partial \gamma} > 0$ and $\frac{\partial \Theta_{\pi a}}{\partial \gamma} > 0$. The response of the natural output gap (16) and inflation (18) to a TFP shock are *smaller in absolute terms* if the externality is more severe, see the red lines in the lower panel of Figure 4. This decline in macroeconomic volatility is associated with the dampening effect that the emission externality exerts on the aggregate production function. The efficient output gap is increasing in γ, since both the natural output gap and the intercept in equation (17) increase in γ. The intercept term from (17) is reflected by the difference between the green and red line. In the simplified three-equation model, the efficient output gap turns positive for $\gamma > 0.08$.

4 Optimal Monetary Policy: LQ-Approach

Having discussed how pro-cyclical emissions affect the competitive equilibrium allocation and the efficient allocation, we now analyze optimal monetary policy. We derive the central bank objective analytically by extending the methodology outlined in Benigno and Woodford (2005) for the case of inefficient competitive equilibria under flexible prices. Since the loss function is available in closed form and easily interpretable, we can take a linear-quadratic approach to analytically show the optimal response of monetary policy to i.i.d. productivity shocks.

4.1 Central Bank Loss Function

The central bank objective function is derived from first principles, i.e. we maximize a utilitarian welfare maximization problem which is closely linked to the distinction between the efficient and natural output gap described in Proposition 1. Since over-production in the competitive equilibrium allocation is at the heart of the mechanism, we follow Benigno and Woodford (2005) and consider the general case where the steady-state level of output and labor are above their efficient levels.

Proposition 4. A second order approximation of the welfare function around the distorted steady state yields the following quadratic loss function:

$$\mathcal{L} = -\mathbb{E}_0\left[\sum_{t=0}^{\infty}\beta^t \frac{U_t - U}{U_c C}\right] \approx \frac{1}{2}\mathbb{E}_0\left[\sum_{t=0}^{\infty}\beta^t\left\{\pi_t^2 + \omega_x\left(x_t^n - x_t^*\right)^2\right\}\right] + t.i.p. \tag{21}$$

where x_t^* is the target level of the natural output gap and ω_x is the weight on output stabilization:

$$\omega_x = \frac{\kappa}{\epsilon}(\zeta(1+\gamma)+\gamma)\ , \tag{22}$$

$$x_t^* = \frac{\Omega_{xa}}{\Omega_x}a_t = -\frac{1+\varphi}{\zeta}\frac{\gamma}{\zeta(1+\gamma)+\gamma}a_t\ . \tag{23}$$

Proof: see Appendix A.4.

The proof is an extension of Benigno and Woodford (2005) to the case of inefficient competitive equilibria under flexible prices. The planner solution takes this inefficiency into account by internalizing the marginal effect of economic activity on economic damages. It is essential that this enters the second order approximation of the household utility function as an additional variable. If the central bank takes this relationship $\Lambda_t = \exp(-\gamma y_t)$ as given, the resulting loss function prescribes to close inflation and natural output gap at all times. Put differently, monetary policy goes along with the market failure in the flexible price allocation and the loss function inherits the inefficiency. We relegate the analytical steps to Appendix C.

Once this additional term enters the loss function, it introduces several linear terms that have to be taken care of appropriately. In addition to a second order approximation of the New Keynesian Phillips Curve that is necessary due to the potentially distorted steady state, we show that it is also necessary to take a second order approximation of the relationship between economic activity and emission damages. In this case, it is possible to derive the loss function (21) that completely separates business cycle stabilization objectives from distortions in the steady state, analogously to Benigno and Woodford (2005).

To gain intuition behind Proposition 4, it is helpful to first consider the text-book case absent the emission externality. In this case, the target level of the output gap x_t^* collapses to zero: efficient and natural output gap coincide and it is optimal to close the output gap at all times. Furthermore, the weight on the output gap ω_x in the loss function reduces to the familiar expression

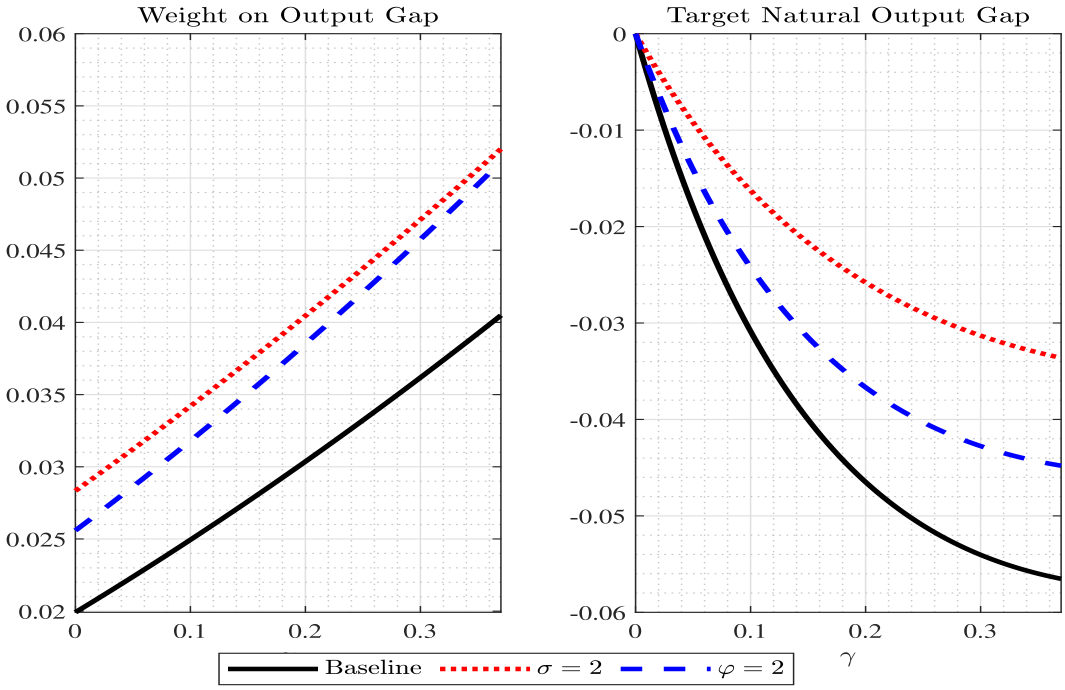
$$\omega_x = \frac{\kappa}{\epsilon}(\sigma + \varphi) \; ,$$

where the auxiliary parameter $\kappa = \frac{(1-\theta\beta)(1-\theta)}{\theta}$ decreases in the share θ of firms that can not adjust prices. A high κ reflects mild nominal rigidities and comparatively large inflation responses to shocks, such that the weight on output stabilization increases in κ.

With $\gamma > 0$, the central bank places a higher weight on output stabilization if the externality is more severe, which is indicated by the left panel of Figure 5. The right panel of Figure 5 illustrates that the target level of the natural output gap is negatively related to the externality parameter γ. The solid black line reflects the baseline parameterization with log-utility and a labor disutility curvature of $\varphi = 4/3$. Unsurprisingly, the weight on the output gap increases while its target level declines in absolute terms if the gains of macroeconomic stabilization increase, either by increasing risk aversion over consumption ($\sigma = 2$, dotted red line) or by increasing the curvature of labor disutility ($\varphi = 2$, dashed blue line). Note that, as in Benigno and Woodford (2005), the steady state wedge does not affect the conduct of macroeconomic stabilization policy, which is solely concerned with addressing the dynamic inefficiency associated with pro-cylical emissions.

Economically, the relationship between weight on the natural output gap ω_x and its target level x_t^* follows directly from the dynamic inefficiency of the competitive equilibrium induced by the emission externality. Production overreacts to a technology shock, relative to the efficient allocation, and the degree of over-

Figure 5: Target level and weight on x_t^n as functions of γ



production is positively relate to severity of the emission externality. The central bank then optimally takes this dynamic inefficiency into account by placing a higher weight on output stabilization. This finding is conceptually related to the case of inefficiently low output due to monopolistic distortions discussed in Clarida, Galí, and Gertler (1999). If efficient output exceeds its natural level, it is optimal to place a higher weight on inflation. Since output is inefficiently high in our analysis, the opposite result emerges.

4.2 Optimal Monetary Policy with i.i.d. Shocks

Next, we characterize optimal monetary policy with i.i.d. shocks to TFP. This corresponds to minimizing the loss function derived in Proposition 4 subject to the NKPC, which can be solved for in closed form.

Proposition 5. If TFP shocks are i.i.d. and carbon taxes are time-invariant

$(\widehat{\tau}_t^c = 0)$ optimal monetary policy is characterized by

$$\pi_t^o = \frac{\omega_x \kappa \zeta}{\kappa^2 \zeta^2 + \omega_x} x_t^* = -\frac{1+\varphi}{\zeta} \frac{\gamma}{\zeta(1+\gamma)+\gamma} \frac{\omega_x \kappa \zeta}{\kappa^2 \zeta^2 + \omega_x} a_t \ , \tag{24}$$

$$x_t^o = \frac{\omega_x}{\kappa^2 \zeta^2 + \omega_x} x_t^* = -\frac{1+\varphi}{\zeta} \frac{\gamma}{\zeta(1+\gamma)+\gamma} \frac{\omega_x}{\kappa^2 \zeta^2 + \omega_x} a_t \ , \tag{25}$$

$$r_t^o = r_t^n - \frac{\sigma \omega_x}{\zeta^2 \kappa^2 + \omega_x} x_t^* = r_t^n + \frac{1+\varphi}{\zeta} \frac{\gamma}{\zeta(1+\gamma)+\gamma} \frac{\sigma \omega_x}{\zeta^2 \kappa^2 + \omega_x} a_t, \tag{26}$$

where r_t^n is the natural rate of interest implicitly defined through (13).
Proof: The central bank chooses inflation and natural output gap to minimize

$$\min_{\pi_t, x_t^n} \quad \frac{1}{2}\mathbb{E}_0\left[\pi_t^2 + \omega_x (x_t^n - x_t^*)^2\right]$$
$$\text{s.t.} \quad \pi_t = \zeta \kappa x_t^n + \beta \mathbb{E}_t[\pi_{t+1}] \ . \tag{27}$$

Taking FOCs and combining them we get the optimal monetary policy that summarizes the trade-off between the natural output gap x_t^n and inflation π_t:

$$\pi_t = -\frac{\omega_x}{\zeta \kappa}(x_t^n - x_t^*) \ . \tag{28}$$

Under i.i.d. shocks, $E_t[\pi_{t+1}] = E_t[x_{t+1}^n] = 0$, so that, plugging the monetary policy rule equation (28) into the NK Phillips curve equation (27), we can re-arrange for the optimal output reaction x_t^o in equation (25). Using this together with the optimal inflation reaction π_t^o from (24) in the IS curve equation (13) and solving for the optimal monetary policy rate r_t^o, we get equation (26). □

Proposition 5 describes how the central bank optimally resolves the trade-off between addressing overproduction and replicating the flexible price allocation over the business cycle. Optimal inflation is negative in response to a positive TFP shock, which can be seen directly from equation (24) and is shown graphically in the left panel of Figure 6. It increases in absolute terms as the externality becomes more severe. If the benefits of business cycle stabilization are increased, either through higher consumption CRRA or higher curvature in labor disutility, the optimal inflation response is muted, ceteris paribus. Equation (25) reveals that the optimal (natural) output gap is a linear combination of the target level x_t^* and the flexible price allocation $(x_t^n = 0)$.

As shown by equation (26), monetary policy does not track the natural interest rate. Instead there is a positive emission adjustment term in equation (26) and the central bank implements a smaller interest rate cut in response to a positive TFP shock than in a model without emission externalities. The adjustment term increases in γ, as suggested in the right panel of Figure 6.

Figure 6: Optimal inflation and interest rate adjustment as functions of γ

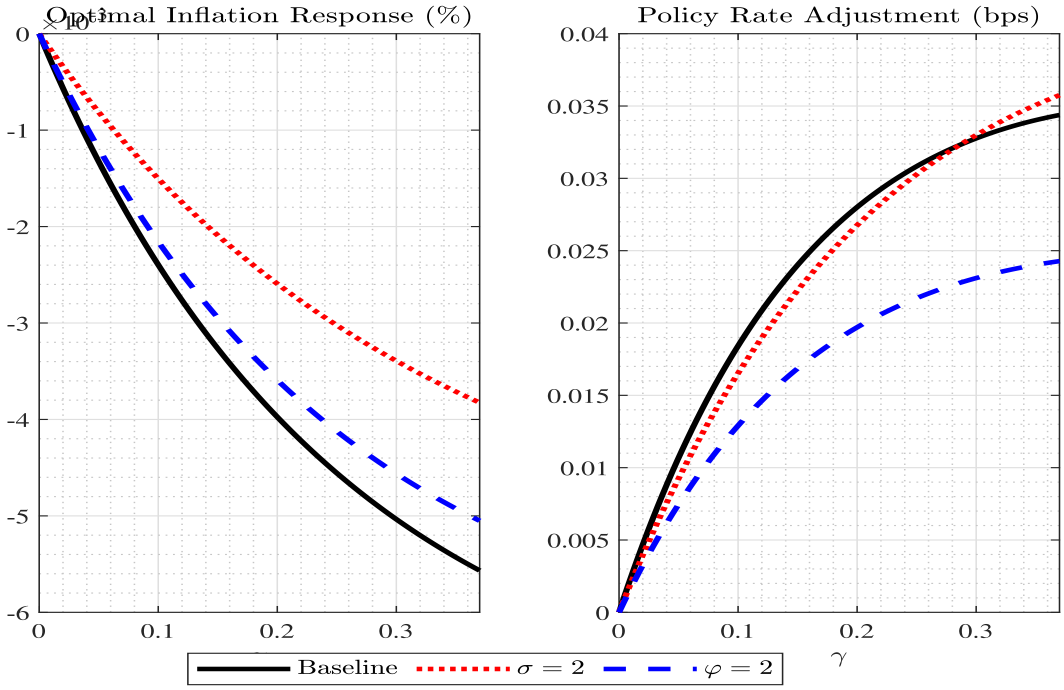
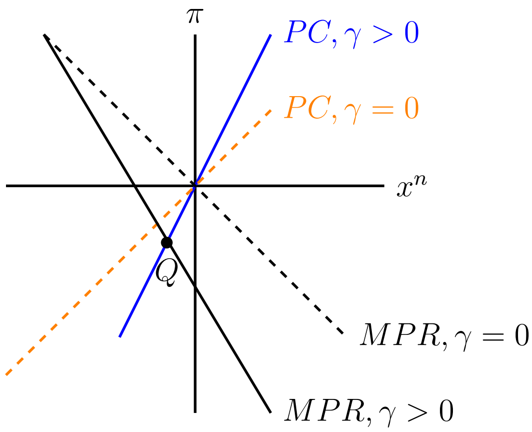


Figure 7 summarizes the effect of emission externalities in a Phillips curve - monetary policy rule diagram. The dashed yellow line refers to the Phillips curve in the textbook New Keynesian model. TFP shocks move the economy on the Phillips curve, since marginal costs decrease after a positive productivity shocks. Firms find it optimal to decrease prices, generating dis-inflationary pressure. Due to the nominal rigidity, not all firms are able to reduce their prices, so that they face a decreased demand for their goods. After a positive TFP shock, output then increases by less than the natural and efficient level, which coincide in the standard model. Consequently, there is downward pressure on the output gap x_t^n. Since it is optimal (and feasible) to perfectly stabilize inflation and both output gaps, i.e. the dashed black MPR crosses the origin, monetary policy can move the economy back to the origin: divine coincidence holds.

The solid blue line refers to the Phillips curve in the case with $\gamma > 0$. From (27), we have seen that the emission externality induces a steepening of the Phillips curve. Holding monetary policy constant, the inflation response is still negative, but larger, which follows directly from Proposition 3. The natural output gap response under constant monetary is also still negative. Importantly, the Phillips curve does cross the origin, so it would be feasible to close the natural output gap and inflation simultaneously. However, the monetary policy rule (solid black line) indicates that this is no longer optimal. The central bank trades off some

Figure 7: Phillips Curve - Monetary Policy Rule Diagram



dis-inflation against maintaining a negative output gap, represented by the point Q. Since the monetary policy rule is also steeper than in the text book model, due to the larger weight on output stabilization, optimal natural output gap and inflation are even further away from zero.

5 Optimal Monetary Policy: Quantitative Results

In the previous section, we have taken a linear-quadratic approach to optimal monetary policy under discretion and discussed how the severity of nominal rigidities and emission externalities determines how close monetary policy comes to implementing the efficient allocation. The analytical characterization of the optimal interest rate response is only available for i.i.d. productivity shocks in the three-equation model where pollution damages that depend on the flow of emissions. In this section, we show numerically that these results carry over to the case of persistent productivity shocks in the model with a fourth equation - the law of motion for emissions. We also demonstrate that the optimal monetary policy response in the four-equation model differs from the textbook model in a quantitatively relevant way.

5.1 Impulse Response Functions

The left column of Figure 8 refers to the simplified three-equation model while the right column represents the baseline four-equation model with persistent emissions. In the standard model without emission externalities, output would increase by one percent in response to a positive one-standard deviation TFP shock. Compared to this benchmark, the economy exhibits a much smaller expansion for the three-equation case with $\delta = 1$, due to the large effect of current output on contemporaneous damages. Output expands merely by around 0.75%. As the right panel shows, the four-equation model behaves much more similar to the standard model without emission externalities. The reason for this is that current emissions have only a very small negative effect on current productivity. In contrast, natural and efficient output gap have a remarkably similar shape in both cases.

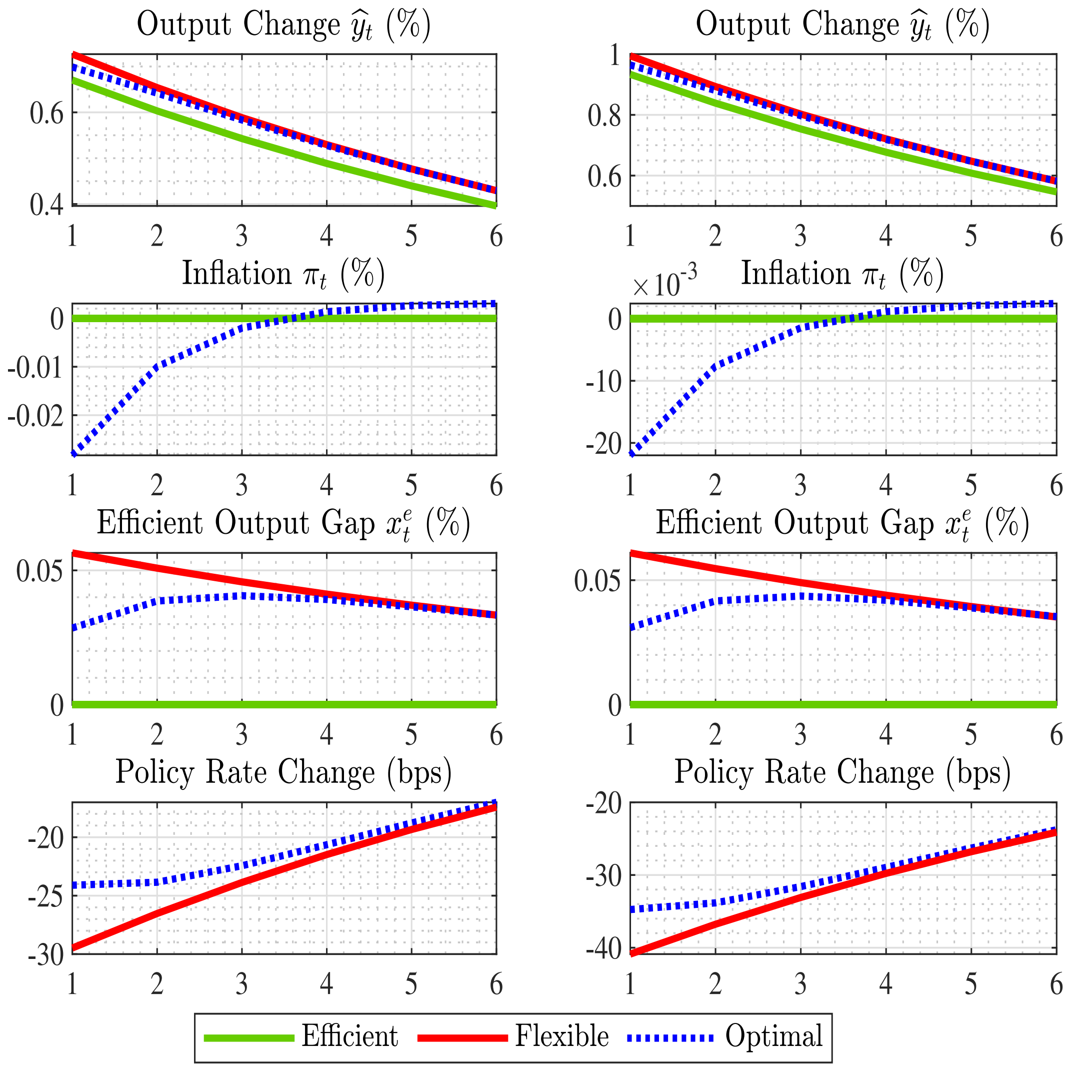
The efficient output response is indicated by the solid green line, which we have already seen in Figure 3. The output expansion under optimal monetary policy for the sticky price model with $\theta = 0.8$ is shown by the dotted blue line. Furthermore, the central bank does not optimally stabilize prices, such that inflation responds negative to an expansionary TFP shock, shown in the second row of Figure 8. More generally, for any θ, optimal monetary policy implements an output expansion between natural and efficient level and never expands output by less than the efficient allocation. This sharply contrasts with the case of exogenous monetary policy display in Figure 3. Intuitively, consider a choice between increasing output by 0.01 percentage points more or less than the efficient allocation, i.e. the green line in the top panel of Figure 8. Loosely speaking, both choices imply the same deviation from first best. However, increasing output by less than the efficient allocation has a negative effect on price stabilization, as it implies a larger deviation from the flexible price equilibrium.

Consequently, the third row of Figure 8 reveals that the efficient output gap is always positive. Furthermore, the optimal interest rate cut is smaller, in absolute terms, than what would be necessary to implement the flexible price equilibrium. Comparing the baseline four-equation to the simplified three-equation model, it turns out that both economies differ in their natural rate, but the discrepancy between the natural rate and the optimal monetary policy rate (the adjustment term in (26)) is quite similar at around 6 basis points. For large shock realizations, the discrepancy between natural and optimal rate is quantitatively relevant, given that the typical monetary policy rate notch is 25 basis points.

5.2 Macro and Welfare Effects

Lastly, we explore the quantitative implications of optimal monetary policy in the presence of socially harmful emissions on macroeconomic aggregates and welfare. We focus on the case without carbon taxes in the main text and show in Appendix B that the results are very similar if the steady state tax is efficient. Table 2 displays key macro moments under Ramsey optimal policy for different values of the Calvo parameter θ in the textbook model (left two columns), the simplified model where emission damages depend on the emission flow (middle columns) and

Figure 8: IRF to TFP-Shock: Optimal Monetary Policy



Notes: Impulse response to a positive one standard deviation shock to TFP, using a second order approximation around the deterministic steady state. The left column refers to the three-equation model where damages depend on the flow of emissions while the right column refers to the four-equation model where damages depend on a persistent stock of emissions.

the baseline model where damages depend on the persistent emission stock (right two columns). Welfare is defined recursively through

$$V_t = \log(c_t) - \frac{n_t^{1+\varphi}}{1+\varphi} + \beta \mathbb{E}_t[V_{t+1}] \ . \tag{29}$$

The left panel of Table 2 considers the counterfactual standard model without

Table 2: Optimal Monetary Policy: Macro and Welfare Effects

	$\gamma = 0$		$\gamma = 0.37$			
			$\delta = 1$		$\delta = 0.0035$	
Calvo Parameter θ	0.6	0.8	0.6	0.8	0.6	0.8
Volatility						
Output Dev. $\widehat{y}_t$ (%)	2.29	2.29	1.67	1.66	2.27	2.25
Eff. Output Gap x_t^e (%)	0	0	0.08	0.07	0.13	0.11
Nat. Output Gap x_t^n (%)	0	0	0.01	0.03	0.01	0.03
Inflation (%)	0	0	0.05	0.03	0.04	0.02
Policy Rate (bps)	93	93	65	63	91	88
Welfare Effect (CE, %)						
Gain of Optimal MP	0.03	0.08	<0.01	0.03	0.01	0.05

Notes: All moments are computed under optimal monetary policy in the presence of a constant carbon tax that renders the deterministic steady state efficient. Inflation volatility is annualized an expressed in percentage points, the policy rate is annualized and expressed in basis points. We express the welfare gain of optimal monetary policy in consumption equivalents $gain^{CE,opt} \equiv \exp\{(1-\beta)(V^{opt} - V^{base})\} - 1$, where V^{base} refers to welfare (29) in an economy where the central bank follows a simple monetary policy rule (15) with inflation coefficient $\phi = 1.5$.

the emission externality. Irrespective of the Calvo parameter, monetary policy can implement the efficient allocation by tracking the natural rate of interest. The associated volatility of the natural interest rate is 93 basis points. Inflation, efficient and natural output gap are zero in all states and the volatility of output is coinciding with a real business cycle model. The welfare gain relative to the case where monetary policy follows a simple Taylor rule (15) with coefficient $\phi = 1.5$ is larger if nominal rigidities are more severe: it increases from 0.03% (in consumption equivalents) to 0.08%. To put the small welfare effects of macroeconomic stabilization policies into perspective, note that they are small in all representative agent models of the business cycle. For instance, the welfare cost of business cycles are merely 0.004% in this model.

The middle panel considers the baseline case with emission damages being proportional to current output, i.e. the case that we studied analytically in the previous section. Macroeconomic volatility is considerably smaller, which we have shown formally in Proposition 3: the standard deviation of output declines from 2.29% for $\theta = 0.6$ to 1.67%. Since monetary policy can not implement the efficient allocation, inflation and both output gaps exhibit a positive volatility. The central bank achieves this by reacting less aggressively to TFP shocks, implying a smaller interest rate volatility of 65bps and 63bps for $\theta = 0.6$ and 0.8, respectively. This also represents the smaller volatility of the natural interest rate in this economy. For damages depending on highly persistent emissions, the interest rate volatility is also smaller (91bs and 88bps for $\theta = 0.6$ and 0.8, respectively), but much closer to the standard model.

Notably, the volatility of efficient output gap is smaller under a higher Calvo parameter. As stressed in Proposition 5 for the case of i.i.d. shocks, more rigid price enable the central bank to reduce the efficient output gap. Correspondingly, the natural output gap is more volatile for $\theta = 0.8$. The welfare gain of optimal monetary policy relative to using a Taylor rule with $\phi = 1.5$, is less than half the size compared to the standard model, as the central bank addresses two dynamic inefficiencies with one instrument.

6 Conclusion

In this paper, we explore the interactions between dynamic real externalities, such as pro-cyclical greenhouse gas emissions and nominal rigidities in a New Keynesian model and study its implications for optimal monetary policy. At the heart of our analysis is the over-reaction of the flexible price equilibrium to productivity shocks that counteracts the typical under-reaction of the sticky price equilibrium in the New Keynesian model. We solve for optimal monetary policy as a second-best solution to a welfare maximization problem. We demonstrate that an additional condition is necessary to ensure that the central bank appropriately internalizes the dynamic inefficiency. This condition requires an extra second order approximation in order to facilitate a clean separation between business cycle stabilization objectives and steady state distortions. We thereby extend the analysis of Benigno and Woodford (2005) to inefficient competitive equilibria.

Building on this methodological contribution, we uncover two main analytical results. First, closing the natural output gap is not optimal from a utilitarian welfare perspective, even though this would be feasible: divine coincidence is broken even for TFP shocks. Second, to tackle this dynamic inefficiency, the central bank optimally targets a non-zero natural output gap, which implies that the optimal inflation volatility is unambiguously larger than in the absence of emission externalities. Quantitatively, the optimal monetary policy response differs by around 6 basis points from the optimal response in a counterfactual economy without pro-cyclical emissions. These results also hold in a larger model with persistent emissions and shocks.

There is evidence that socially harmful emissions also have a direct effect on macroeconomic volatility and inflation through a disaster risk channel or through commodity and energy price volatility, from which we abstract in our analysis.

Furthermore, carbon taxation can also induce inflation by increasing energy prices, which has been subject to recent discussions about targeting core and headline inflation. Exploring the interactions between these additional channels, nominal rigidities, and its implications for monetary policy is left for future research.

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A Proofs

This section contains all proofs omitted in the main text.

A.1 Proof of Proposition 1

The aggregate production function can be written as $y_t = A_t \Lambda_t n_t$, while the goods market clearing condition is given by $y_t = c_t$.

Efficient Allocation The planner problem is

$$\max_{c_t, n_t, y_t, \Lambda_t, E_t} \quad \sum_{t=0}^{\infty} \beta^t \left[\frac{c_t^{1-\sigma}}{1-\sigma} - \frac{n_t^{1+\varphi}}{1+\varphi} \right] \quad \text{s.t.}$$

$$c_t = y_t \ , \qquad (\lambda_t)$$

$$y_t = A_t \Lambda_t n_t \ , \qquad (\mu_t)$$

$$\Lambda_t = \exp\left\{ -\gamma \frac{E_t}{E} \right\} \ , \qquad (\nu_t)$$

$$E_t = y_t + (1-\delta) E_{t-1} \ . \qquad (\psi_t)$$

Setting up the Lagrangian

$$\max_{c_t, n_t, y_t, \Lambda_t, E_t} \sum_{t=0}^{\infty} \beta^t \Bigg[\frac{c_t^{1-\sigma}}{1-\sigma} - \frac{n_t^{1+\varphi}}{1+\varphi} + \lambda_t \Big(y_t - c_t \Big) + \mu_t \Big(A_t \Lambda_t n_t - y_t \Big) + \nu_t \Big(\exp\left\{ -\gamma \frac{E_t}{E} \right\} - \Lambda_t \Big) + \psi_t \Big(E_t - y_t - (1-\delta) E_{t-1} \Big) \Bigg]$$

and taking FOCs yields

$$\lambda_t = c_t^{-\sigma} \qquad \text{(A.1)}$$

$$\mu_t A_t \Lambda_t = n_t^{\varphi} \qquad \text{(A.2)}$$

$$\lambda_t - \mu_t - \psi_t = 0 \qquad \text{(A.3)}$$

$$\mu_t A_t n_t = \nu_t \qquad \text{(A.4)}$$

$$-\nu_t \frac{\gamma}{E} \Lambda_t + \psi_t - \beta(1-\delta)\psi_{t+1} = 0 \qquad \text{(A.5)}$$

Combining (A.4) and (A.5):

$$\psi_t = \beta(1-\delta)\psi_{t+1} + \mu_t \frac{\gamma}{E} y_t$$

With (A.3):

$$\lambda_t = \beta(1-\delta)\big(\lambda_{t+1} - \mu_{t+1}\big) + \mu_t \frac{\gamma}{E} y_t + \mu_t$$

With $\delta = 1$, we have $E = y$ and this expression collapses to the expression for MPN^e that we have derived in the main text. With $\delta < 1$, we can re-arrange to

$$\lambda_t - \mu_t\big(1 + \frac{\gamma}{E} y_t\big) = \beta(1-\delta)\big(\lambda_{t+1} - \mu_{t+1}\big)$$

Using (A.1) and (A.2):

$$c_t^{-\sigma} - \frac{n_t^{\varphi}}{A_t \Lambda_t}\big(1 + \frac{\gamma}{E} y_t\big) = \beta(1-\delta)\big(c_{t+1}^{-\sigma} - \frac{n_{t+1}^{\varphi}}{A_{t+1}\Lambda_{t+1}}\big) \tag{A.6}$$

Exploiting $c_t = y_t = A_t \Lambda_t n_t$, the efficient output is given by the solution y_t^e to:

$$y_t^{-\sigma} - \frac{y_t^{\varphi}}{(A_t \Lambda_t)^{1+\varphi}}\big(1 + \frac{\gamma}{E} y_t\big) = \beta(1-\delta)\big(y_{t+1}^{-\sigma} - \frac{y_{t+1}^{\varphi}}{(A_{t+1}\Lambda_{t+1})^{1+\varphi}}\big) \ . \tag{A.7}$$

Competitive Equilibrium Next, we derive the natural level of output consistent with flexible prices. The relevant equilibrium conditions are:

- Euler equation and labor supply condition,

$$c_t^{-\sigma} = \beta r_t^s \mathbb{E}_t \left[\frac{c_{t+1}^{-\sigma}}{\pi_{t+1}} \right] \ , \tag{A.8}$$

$$n_t^{\varphi} = w_t c_t^{-\sigma} \ . \tag{A.9}$$

- emission damage function:

$$\Lambda_t = \exp\Big(-\gamma \frac{E_t}{E} \Big) \ , \tag{A.10}$$

- aggregate production function:

$$\Delta_t y_t = A_t \Lambda_t n_t \ , \tag{A.11}$$

  where Δ_t is the price dispersion.

- labor demand:

$$(1 - \tau^n) w_t = mc_t A_t \Lambda_t \ , \tag{A.12}$$

- good market clearing:

$$y_t = c_t \ . \tag{A.13}$$

- Optimal pricing

$$p_t^* = \frac{\mu}{1-\tau_t^c}\frac{\xi_{1,t}}{\xi_{2,t}} \ , \tag{A.14}$$

where $\mu \equiv \frac{\epsilon}{\epsilon-1}$ and

$$\xi_{1,t} = mc_t y_t + \beta\theta \frac{c_{t+1}^{-\sigma}}{c_t^{-\sigma}} \pi_{t+1}^{\epsilon} \xi_{1,t+1} \ , \tag{A.15}$$

$$\xi_{2,t} = y_t + \beta\theta \frac{c_{t+1}^{-\sigma}}{c_t^{-\sigma}} \pi_{t+1}^{\epsilon-1} \xi_{2,t+1} \ . \tag{A.16}$$

- Inflation:

$$1 = (1-\theta)(p_t^*)^{1-\epsilon} + \theta\pi_t^{\epsilon-1} \ . \tag{A.17}$$

- Price dispersion:

$$\Delta_t = (1-\theta)(p_t^*)^{-\epsilon} + \theta\pi_t^{\epsilon}\Delta_{t-1} \ . \tag{A.18}$$

Equations (A.8) to (A.18) characterize the competitive equilibrium for the eleven endogenous variables $\{\Delta_t, y_t, \Lambda_t, n_t, w_t, mc_t, c_t, p_t^*, \xi_{1,t}, \xi_{2,t}.\pi_t\}$.

If prices are flexible, we normalize the price level such that $\Delta_t = \pi_t = p_t^* = 1$ and $\frac{\xi_{1,t}}{\xi_{2,t}} = mc_t$. In this case, (A.14) simplifies to $p_t^* = \frac{\mu}{(1-\tau_t^c)} mc_t$. Marginal costs are simply given by $mc_t = \frac{(1-\tau^n)w_t}{A_t\Lambda_t}$. If the labor subsidy $\tau^n = \frac{1}{\epsilon}$ corrects for the steady state monopolistic distortion, using equations (A.12) and (A.9) gives:

$$1 = \frac{\mu}{1-\tau_t^c} mc_t = \frac{1}{1-\tau_t^c} \underbrace{\mu(1-\tau^n)}_{=1} \frac{w_t}{A_t\Lambda_t} = \frac{n_t^{\varphi} c_t^{\sigma}}{(1-\tau_t^c)A_t\Lambda_t} \ . \tag{A.19}$$

Using goods market clearing (A.13) and the production function (A.11) to eliminate c_t and n_t, we obtain

$$1 = \frac{y_t^{\sigma+\varphi}}{(1-\tau_t^c)(A_t\Lambda_t)^{1+\varphi}} \ .$$

Solving for y_t yields the natural output level (6). □

A.2 Proof of Lemma 2

Log-linearizing the natural output level (6) around the deterministic steady state gives

$$(\sigma+\varphi)\widehat{y}_t^n = (1+\varphi)a_t - (1+\varphi)\gamma\widehat{y}_t^n - \frac{\tau^c}{1-\tau^c}\widehat{\tau}_t^c \ ,$$

where hats indicate log-deviations from steady state. Re-arranging for $\widehat{y}_t^n$ yields equation (10). Log-linearizing the efficient output level (8) and noticing that $\hat{\lambda}_t = -\gamma\hat{y}_t$ yields

$$\begin{aligned} & (\sigma+\varphi)\widehat{y}_t^e = (1+\varphi)a_t - (1+\varphi)\gamma\frac{y^e}{y}\widehat{y}_t^e - \frac{\gamma}{1+\gamma}\widehat{y}_t^e \\ \Leftrightarrow \quad & \left[\sigma+\varphi+(1+\varphi)\gamma\frac{y^e}{y}+\frac{\gamma}{1+\gamma}\right]\widehat{y}_t^e = (1+\varphi)a_t \ . \end{aligned} \tag{A.20}$$

Re-arranging for $\widehat{y}_t^e$, we arrive at equation (11). □

A.3 Proof of Proposition 2

The log-linearized equilibrium conditions in the simplified model ($\delta = 1$) are given by

- Euler equation (A.8):

$$\sigma\widehat{c}_t = \sigma\widehat{c}_{t+1} - (\widehat{r}_t^s - \widehat{\pi}_{t+1}) \ . \tag{A.21}$$

- Optimal labor supply (A.9):

$$\widehat{w}_t = \varphi\widehat{n}_t + \sigma\widehat{c}_t \ . \tag{A.22}$$

- Emission damages (A.10):

$$\widehat{\Lambda}_t = -\gamma\widehat{y}_t \ . \tag{A.23}$$

- Production function (A.11):

$$\widehat{\Delta}_t + \widehat{y}_t = a_t - \gamma\widehat{y}_t + \widehat{n}_t \ . \tag{A.24}$$

- Labor demand (A.12):

$$\widehat{w}_t = \widehat{mc}_t + a_t - \gamma\widehat{y}_t \ . \tag{A.25}$$

- Optimal pricing (A.14), (A.15), and (A.16):

$$\widehat{p}_t^* = \frac{\tau^c}{1-\tau^c}\widehat{\tau}_t^c + \widehat{\xi}_{1t} - \widehat{\xi}_{2t} \tag{A.26}$$

$$\widehat{\xi_{1,t}} = (1-\theta\beta)\widehat{mc}_t + (1-\theta\beta)\widehat{y}_t - \theta\beta\sigma\widehat{c}_{t+1} + \theta\beta\sigma\widehat{c}_t + \epsilon\theta\beta\widehat{\pi}_{t+1} + \theta\beta\widehat{\xi}_{1,t+1} \tag{A.27}$$

$$\widehat{\xi_{2,t}} = (1-\theta\beta)\widehat{y}_t - \theta\beta\sigma\widehat{c}_{t+1} + \theta\beta\sigma\widehat{c}_t + (\epsilon-1)\theta\beta\widehat{\pi}_{t+1} + \theta\beta\widehat{\xi}_{2,t+1} \ , \tag{A.28}$$

where we assumed no steady state inflation, $\overline{\pi} = 1$.

- Inflation (A.17):

$$0 = (1-\epsilon)(1-\theta)\widehat{p}_t^* + \theta(\epsilon-1)\widehat{\pi}_t \ \Leftrightarrow \ \widehat{p}_t^* = \frac{\theta}{1-\theta}\widehat{\pi}_t \ . \tag{A.29}$$

- Price dispersion (A.18):

$$\widehat{\Delta}_t = -\epsilon(1-\theta)\widehat{p}_t^* + \theta\epsilon\widehat{\pi}_t + \theta\widehat{\Delta}_{t-1} \ \Leftrightarrow \ \widehat{\Delta}_t = \theta\widehat{\Delta}_{t-1} \Leftrightarrow \ \widehat{\Delta}_t = 0 \ .$$

- Market clearing (A.13):

$$\widehat{c}_t = \widehat{y}_t \ . \tag{A.30}$$

As before, we define the natural output gap as

$$x_t^n = \widehat{y}_t - \widehat{y}_t^n = \widehat{y}_t - \frac{1}{\zeta}\left[(1+\varphi)a_t - \frac{\tau^c}{1-\tau^c}\widehat{\tau}_t^c\right] ,$$

and efficient output gap:

$$x_t^e = \widehat{y}_t - \widehat{y}_t^e = \widehat{y}_t - \frac{1}{\widetilde{\zeta}+\widetilde{\gamma}}\left[(1+\varphi)a_t\right] .$$

We first derive the log-linearized NKPC. Subtracting the auxiliary terms in the optimal pricing condition, (A.28) and (A.27), from each other, we have

$$\widehat{\xi}_{1t} - \widehat{\xi}_{2t} = (1-\theta\beta)\widehat{mc}_t + \theta\beta\pi_{t+1} + \theta\beta(\widehat{\xi}_{1,t+1} - \widehat{\xi}_{2,t+1}) \ .$$

Plugging this condition and the definition of inflation (A.29) into the expression for the optimal price (A.26), we get an expression for marginal costs:

$$\frac{\theta}{1-\theta}\pi_t = \frac{\tau^c}{1-\tau^c}\widehat{\tau}_t^c + (1-\theta\beta)\widehat{mc}_t + \theta\beta\left(\pi_{t+1} + \frac{\theta}{1-\theta}\pi_{t+1} - \frac{\tau^c}{1-\tau^c}\widehat{\tau}_{t+1}^c\right)$$

$$\Leftrightarrow \pi_t = \underbrace{\frac{(1-\theta\beta)(1-\theta)}{\theta}}_{=\kappa}\widehat{mc}_t + \beta\pi_{t+1} + \frac{1-\theta}{\theta}\frac{\tau^c}{1-\tau^c}\left(\widehat{\tau}_t - \theta\beta\widehat{\tau}_{t+1}^c\right) . \tag{A.31}$$

Combining labor supply (A.22), production function (A.24), and labor demand (A.25), we get a second condition linking output to marginal costs:

$$\widehat{mc}_t = \widehat{w}_t - a_t + \gamma\widehat{y}_t = \varphi\widehat{n}_t + \sigma\widehat{c}_t - a_t + \gamma\widehat{y}_t = \varphi(\widehat{y}_t - a_t + \gamma\widehat{y}_t) + \sigma\widehat{y}_t - a_t - \gamma\widehat{y}_t$$
$$= \underbrace{[\sigma + \varphi + (1+\varphi)\gamma]}_{=\zeta}\widehat{y}_t - (1+\varphi)a_t \ .$$

Plugging this condition into equation (A.31):

$$\pi_t = \kappa\zeta\left[\underbrace{y_t - \frac{1+\varphi}{\zeta}a_t + \frac{1}{\zeta}\frac{\tau^c}{1-\tau^c}\widehat{\tau}^c_t}_{=x^n_t} - \frac{1}{\zeta}\frac{\tau^c}{1-\tau^c}\widehat{\tau}^c_t\right] + \beta\pi_{t+1} + \frac{1-\theta}{\theta}\frac{\tau^c}{1-\tau^c}\left(\widehat{\tau}^c_t - \theta\beta\widehat{\tau}^c_{t+1}\right)$$
$$= \kappa\zeta x^n_t + \beta\pi_{t+1} - \kappa\frac{\tau^c}{1-\tau^c}\widehat{\tau}^c_t + \frac{\kappa}{1-\theta\beta}\frac{\tau^c}{1-\tau^c}\widehat{\tau}^c_t - (1-\theta)\beta\frac{\tau^c}{1-\tau^c}\widehat{\tau}^c_{t+1}$$
$$= \kappa\zeta x^n_t + \beta\pi_{t+1} + (1-\theta)\beta\frac{\tau^c}{1-\tau_c}(\widehat{\tau}^c_t - \widehat{\tau}^c_{t+1}) \ ,$$

which is the NKPC (14) in Proposition 2.
To get the dynamic IS-equation (13), start from the linearized Euler equation equation (A.21) and impose market clearing (A.30) to get:

$$\widehat{y}_t = \widehat{y}_{t+1} - \frac{1}{\sigma}(r^s_t - \pi_{t+1}) \ \Leftrightarrow$$
$$\widehat{y}_t - \frac{1}{\zeta}\left[(1+\varphi)a_t - \frac{\tau^c}{1-\tau^c}\widehat{\tau}^c_t\right] + \frac{1}{\zeta}\left[(1+\varphi)a_t - \frac{\tau^c}{1-\tau^c}\widehat{\tau}^c_t\right] =$$
$$\widehat{y}_{t+1} - \frac{1}{\zeta}\left[(1+\varphi)a_{t+1} - \frac{\tau^c}{1-\tau^c}\widehat{\tau}^c_{t+1}\right] + \frac{1}{\zeta}\left[(1+\varphi)a_{t+1} - \frac{\tau^c}{1-\tau^c}\widehat{\tau}^c_{t+1}\right] - \frac{1}{\sigma}(r^s_t - \pi_{t+1}) \ \Leftrightarrow$$
$$x^n_t + \frac{1}{\zeta}\left[(1+\varphi)a_t - \frac{\tau^c}{1-\tau^c}\widehat{\tau}^c_t\right] = x^n_{t+1} + \frac{1}{\zeta}\left[(1+\varphi)a_{t+1} - \frac{\tau^c}{1-\tau^c}\widehat{\tau}^c_{t+1}\right] - \frac{1}{\sigma}(r^s_t - \pi_{t+1}) \ \Leftrightarrow$$
$$x^n_t = x^n_{t+1} - \frac{1}{\sigma}(r^s_t - \pi_{t+1}) + \frac{1}{\zeta}\left[(1+\varphi)(a_{t+1} - a_t) - \frac{\tau^c}{1-\tau^c}(\widehat{\tau}^c_{t+1} - \widehat{\tau}^c_t)\right]$$

□

A.4 Proof of Proposition 4

As a first step, we take a **second order approximation** of the welfare objective, i.e. households period utility function U_t:

$$U_t - U \approx c^{1-\sigma}\left\{\frac{c_t - c}{c} - \frac{\sigma}{2}\left(\frac{c_t - c}{c}\right)^2 - \frac{n^{1+\varphi}}{c^{1-\sigma}}\left[\frac{n_t - n}{n} + \frac{\varphi}{2}\left(\frac{n_t - n}{n}\right)^2\right]\right\} .$$

Re-arranging yields

$$\frac{U_t - U}{U_c c} = \frac{U_t - U}{c^{1-\sigma}} \approx \frac{c_t - c}{c} - \frac{\sigma}{2}\left(\frac{c_t - c}{c}\right)^2 - \frac{n^{1+\varphi}}{c^{1-\sigma}}\left[\frac{n_t - n}{n} + \frac{\varphi}{2}\left(\frac{n_t - n}{n}\right)^2\right].$$

For a generic variable x, up to second order, it holds that $\frac{x_t - x}{x} = \widehat{x}_t + \frac{\widehat{x}_t^2}{2}$ with $\widehat{x}_t \equiv \log x_t - \log x$. Also, from the steady state condition for natural output equation (8) and market clearinig, the following condition holds:

$$\frac{n^{1+\varphi}}{c^{1-\sigma}} = y^{1+\varphi}\frac{y^{\sigma-1}}{(A\Lambda)^{1+\varphi}} = \frac{y^{\sigma+\phi}}{(A\Lambda)^{1+\varphi}} = \frac{(1-\tau^c)(A\Lambda)^{1+\varphi}}{(A\Lambda)^{1+\varphi}} = 1 - \tau^c$$

Hence, we can re-write the second order approximation of the welfare function as:

$$\frac{U_t - U}{c^{1-\sigma}} \approx \widehat{c}_t + \frac{\widehat{c}_t^2}{2} - \frac{\sigma}{2}\widehat{c}_t^2 - (1-\tau^c)\left[\widehat{n}_t + \frac{\widehat{n}_t^2}{2} + \frac{\varphi}{2}\widehat{n}_t^2\right].$$

In order to **express the loss function in terms of the output gap** x_t^n and inflation π_t, we first make use of the market clearing condition $\widehat{c}_t = \widehat{y}_t$ and the production function $\widehat{n}_t = \widehat{y}_t + \widehat{\Delta}_t - a_t - \widehat{\Lambda}_t$:

$$\frac{U_t - U}{c^{1-\sigma}} \approx \widehat{y}_t + \frac{1-\sigma}{2}\widehat{y}_t^2 - (1-\tau)\left[\widehat{y}_t + \widehat{\Delta}_t - a_t - \widehat{\Lambda}_t + \frac{1+\varphi}{2}\left(\widehat{y}_t + \widehat{\Delta}_t - a_t - \widehat{\Lambda}_t\right)^2\right]$$

Define $\Phi \equiv (1-\tau)(1+\gamma) - 1$, so that:

$$\frac{U_t - U}{c^{1-\sigma}} \approx \widehat{y}_t + \frac{1-\sigma}{2}\widehat{y}_t^2 - \frac{1+\Phi}{1+\gamma}\left[\widehat{y}_t + \widehat{\Delta}_t - a_t - \widehat{\Lambda}_t + \frac{1+\varphi}{2}\left(\widehat{y}_t + \widehat{\Delta}_t - a_t - \widehat{\Lambda}_t\right)^2\right]$$

The newly defined parameter Φ is a measure of the steady state inefficiency: it takes the maximum value γ if $\tau = 0$ and it decreases to 0 as τ approaches the efficient level $\frac{\gamma}{1+\gamma}$. Throughout the proof, we consider the general case of a sub-optimal steady state carbon tax, $\tau^c \leq \widetilde{\gamma}$, i.e. $\Phi \geq 0$. Eliminating all terms independent of policy and of order higher than two, we then obtain:

$$\begin{aligned}
\frac{U_t - U}{c^{1-\sigma}} \approx & \widehat{y}_t + \frac{1-\sigma}{2}\widehat{y}_t^2 - \frac{1+\Phi}{1+\gamma}\Big[\widehat{y}_t + \widehat{\Delta}_t - \widehat{\Lambda}_t + \frac{1+\varphi}{2}\widehat{y}_t^2 + \frac{1+\varphi}{2}\widehat{\Lambda}_t^2 \\
& - (1+\varphi)\widehat{y}_t a_t - (1+\varphi)\widehat{y}_t\widehat{\Lambda}_t + (1+\varphi)\widehat{\Lambda}_t a_t\Big] + t.i.p. \\
\approx & \frac{\gamma - \Phi}{1+\gamma}\widehat{y}_t + \frac{1+\Phi}{1+\gamma}\widehat{\Lambda}_t - \frac{\widehat{y}_t^2}{2}\left[-1 + \sigma + \frac{1+\Phi}{1+\gamma}(1+\varphi)\right] - \frac{1+\Phi}{1+\gamma}\widehat{\Delta}_t \\
& - \frac{(1+\Phi)(1+\varphi)}{2(1+\gamma)}\widehat{\Lambda}_t^2 + \frac{1+\Phi}{1+\gamma}(1+\varphi)(\widehat{y}_t a_t + \widehat{y}_t\widehat{\Lambda}_t - \widehat{\Lambda}_t a_t) + t.i.p.
\end{aligned}$$

We are then ready to **evaluate the loss function**:

$$\mathcal{L} \equiv -\mathcal{W} \approx \mathbb{E}_0 \left[\sum_{t=0}^{\infty} \beta^t \left\{ \frac{1}{2} \left\{ -1 + \sigma + \frac{1+\Phi}{1+\gamma}(1+\varphi) \right\} y_t^2 + \frac{1+\Phi}{1+\gamma} \widehat{\Delta}_t + \frac{(1+\Phi)(1+\varphi)}{2(1+\gamma)} \Lambda_t^2 - \right.\right.$$
$$\left.\left. - \frac{1+\Phi}{1+\gamma}(1+\varphi)(\widehat{y}_t a_t + \widehat{y}_t \widehat{\Lambda}_t - \widehat{\Lambda}_t a_t) + \frac{\Phi-\gamma}{1+\gamma} \widehat{y}_t - \frac{1+\Phi}{1+\gamma} \widehat{\Lambda}_t \right\} \right] .$$

We first substitute the price dispersion term by an expression related to the NKPC via inflation. The discounted sum of log price dispersions is given by $\sum_{t=0}^{\infty} \beta^t \widehat{\Delta}_t \approx \frac{\epsilon}{2\kappa} \sum_{t=0}^{\infty} \beta^t \pi_t^2$, with the auxiliary parameter $\kappa = \frac{(1-\theta)(1-\theta\beta)}{\theta}$ governing the slope of the NKPC. Therefore, the loss function is given by

$$\mathcal{L} \equiv -\mathcal{W} \approx \mathbb{E}_0 \left[\sum_{t=0}^{\infty} \beta^t \left\{ \frac{1}{2} \left\{ -1 + \sigma + \frac{1+\Phi}{1+\gamma}(1+\varphi) \right\} \widehat{y}_t^2 + \frac{1+\Phi}{1+\gamma} \frac{\epsilon}{2\kappa} \pi_t^2 + \frac{(1+\Phi)(1+\varphi)}{2(1+\gamma)} \widehat{\Lambda}_t^2 \right.\right.$$
$$\left.\left. - \frac{1+\Phi}{1+\gamma}(1+\varphi)(\widehat{y}_t a_t + \widehat{y}_t \widehat{\Lambda}_t - \widehat{\Lambda}_t a_t) + \frac{\Phi-\gamma}{1+\gamma} \widehat{y}_t - \frac{1+\Phi}{1+\gamma} \widehat{\Lambda}_t \right\} \right] . \quad \text{(A.32)}$$

As next step, we **eliminate the linear terms** $\frac{\Phi-\gamma}{1+\gamma} \widehat{y}_t$ and $\frac{1+\Phi}{1+\gamma} \widehat{\Lambda}_t$ in (A.32). To do so, we take a second order approximation of both the optimal pricing condition (A.14) and of the relationship between pollution and economic activity (A.10). Starting from the latter, the second order approximation of eq. (A.10) reads:

$$\widehat{\Lambda}_t + \frac{1}{2} \widehat{\Lambda}_t^2 = -\gamma \widehat{y}_t - \frac{1}{2}\gamma(1-\gamma) \widehat{y}_t^2 \Leftrightarrow \widehat{\Lambda}_t = -\gamma \widehat{y}_t - \frac{1}{2}\gamma(1-\gamma)\widehat{y}_t^2 - \frac{1}{2} \widehat{\Lambda}_t^2 . \quad \text{(A.33)}$$

Plugging this into (A.32), we get:

$$\mathcal{L} \equiv -\mathcal{W} \approx \mathbb{E}_0 \left[\sum_{t=0}^{\infty} \beta^t \left\{ \frac{1}{2} \left\{ -1 + \sigma + \frac{1+\Phi}{1+\gamma}(1+\varphi+\gamma(1-\gamma) \right\} \widehat{y}_t^2 + \frac{1+\Phi}{1+\gamma} \frac{\epsilon}{2\kappa} \pi_t^2 \right.\right.$$
$$\left.\left. + \frac{(1+\Phi)(2+\varphi)}{2(1+\gamma)} \widehat{\Lambda}_t^2 - \frac{1+\Phi}{1+\gamma}(1+\varphi)(\widehat{y}_t a_t + \widehat{y}_t \widehat{\Lambda}_t - \widehat{\Lambda}_t a_t) + \Phi \widehat{y}_t \right\} \right] . \quad \text{(A.34)}$$

For what concerns the pricing condition, to eliminate the linear term Φy_t, we exploit that a second order approximation of eq. (A.14) leads to the following (extended) NKPC:

$$\pi_t + \frac{\epsilon-1}{2(1-\theta)} \pi_t^2 + \frac{1-\theta\beta}{2} G_t \pi_t = \kappa \left[\widehat{\xi}_{1t} - \widehat{\xi}_{2t} + \frac{1}{2}(\widehat{\xi}_{1t}^2 - \widehat{\xi}_{2t}^2) \right] + \beta \pi_{t+1} \quad \text{(A.35)}$$
$$+ \beta \frac{1-\theta\beta}{2} G_{t+1} \pi_{t+1} + \beta \frac{\epsilon-1}{2(1-\theta)} \pi_{t+1}^2 + \beta \frac{\epsilon}{2} \pi_{t+1}^2 .$$

Here, the log-linearized Calvo terms are given by $\widehat{\xi}_{1t} \equiv mc_t - \sigma\widehat{c}_t + \widehat{y}_t$ and $\widehat{\xi}_{2t} \equiv \widehat{y}_t - \sigma\widehat{c}_t$ and the auxiliary term G_t is defined as the present value of future Calvo terms

$$G_t \equiv \sum_{\tau=t}^{\infty}(\theta\beta)^{\tau-t}(\widehat{\xi}_{1,t,\tau} + \widehat{\xi}_{2,t,\tau}) \ ,$$

where $\widehat{\xi}_{1,t,\tau} \equiv \widehat{\xi}_{1\tau} + \epsilon\sum_{s=t+1}^{\tau}\pi_s$ and $\widehat{\xi}_{1,t,\tau} \equiv \xi_{1\tau} + (\epsilon-1)\sum_{s=t+1}^{\tau}\pi_s$. Defining $H_t \equiv \pi_t + \frac{\epsilon-1}{2(1-\theta)}\pi_t^2 + \frac{1-\theta\beta}{2}G_t\pi_t + \frac{\epsilon}{2}\pi_t^2$, the extended NKPC (A.35) can be rewritten as:

$$H_t = \kappa\left[\widehat{\xi}_{1t} - \widehat{\xi}_{2t} + \frac{1}{2}(\widehat{\xi}_{1t}^2 - \widehat{\xi}_{2t}^2)\right] + \beta\frac{\epsilon}{2}\pi_t^2 + \beta H_{t+1} \ .$$

Iterating forward:

$$H_0 = \kappa\mathbb{E}_0\left[\sum_{t=0}^{\infty}\beta^t\left\{\widehat{\xi}_{1t} - \widehat{\xi}_{2t} + \frac{1}{2}(\widehat{\xi}_{1t}^2 - \widehat{\xi}_{2t}^2)\right\}\right] + \frac{\epsilon}{2}\sum_{t=0}^{\infty}\beta^t\pi_t^2 \ . \tag{A.36}$$

Now, using market clearing (A.30) and production function (A.24):

$$\begin{aligned}\widehat{\xi}_{1t} =& \widehat{mc}_t + \widehat{y}_t - \sigma\widehat{c}_t = \widehat{w}_t - a_t - \widehat{\Lambda}_t + \widehat{y}_t - \sigma\widehat{y}_t = \varphi\widehat{n}_t - a_t - \widehat{\Lambda}_t + \widehat{y}_t \\ =& \varphi(\widehat{\Delta}_t + \widehat{y}_t - a_t - \widehat{\Lambda}_t) - a_t - \widehat{\Lambda}_t + \widehat{y}_t = (1+\varphi)\widehat{y}_t - (1+\varphi)a_t - (1+\varphi)\widehat{\Lambda}_t + \varphi\widehat{\Delta}_t \ ,\end{aligned}$$

Hence, the difference between the Calvo terms reduces to:

$$\widehat{\xi}_{1t} - \widehat{\xi}_{2t} \approx (\sigma+\varphi)\widehat{y}_t - (1+\varphi)\widehat{\Lambda}_t + \varphi\widehat{\Delta}_t \ .$$

The difference between the squared Calvo terms, ignoring terms of order higher than two and terms irrelevant for policy, simplifies to:

$$\begin{aligned}\widehat{\xi}_{1t}^2 - \widehat{\xi}_{2t}^2 =& [(1+\varphi)\widehat{y}_t - (1+\varphi)a_t - (1+\varphi)\Lambda_t + \varphi\widehat{\Delta}_t]^2 - (1-\sigma)^2\widehat{y}_t^2 \\ \approx& (\sigma+\varphi)(2+\varphi-\sigma)\widehat{y}_t^2 + (1+\varphi)^2[\widehat{\Lambda}_t^2 - 2\widehat{y}_t a_t - 2\widehat{y}_t\widehat{\Lambda}_t + 2a_t\widehat{\Lambda}_t] \ .\end{aligned}$$

Hence, we can then rewrite the discounted sum H_0 as

$$\begin{aligned}H_0 \approx& \kappa\mathbb{E}_0\left[\sum_{t=0}^{\infty}\beta^t\left\{(\sigma+\varphi)\widehat{y}_t + \varphi\widehat{\Delta}_t - (1+\varphi)\widehat{\Lambda}_t + \frac{1}{2}(\sigma+\varphi)(2+\varphi-\sigma)\widehat{y}_t^2 + \frac{(1+\varphi)^2}{2}\widehat{\Lambda}_t^2 - \right.\right. \\ &\left.\left. - (1+\varphi)^2[\widehat{y}_t a_t + \widehat{y}_t\widehat{\Lambda}_t - a_t\widehat{\Lambda}_t] + \frac{\epsilon}{2\kappa}\pi_t^2\right\}\right] \ .\end{aligned}$$

We plug in the second-order approximation of emission damages (A.33):

$$H_0 \approx \kappa \mathbb{E}_0 \left[\sum_{t=0}^{\infty} \beta^t \left\{ \zeta y_t + \varphi \Delta_t + \frac{1}{2}[(\sigma+\varphi)(2+\varphi-\sigma) + (1+\varphi)\gamma(1-\gamma)]\widehat{y}_t^2 \right.\right.$$
$$\left.\left. + (2+\varphi)\frac{1+\varphi}{2}\widehat{\Lambda}_t^2 - (1+\varphi)^2[\widehat{y}_t a_t + \widehat{y}_t \widehat{\Lambda}_t - a_t \widehat{\Lambda}_t] + \frac{\epsilon}{2\kappa}\pi_t^2 \right\} \right] .$$

Since H_0 is given and the present value of price dispersions is linked to the present value of squared inflation $\sum \beta^t \Delta_t = \frac{\epsilon}{2\kappa} \sum \beta^t \pi_t^2$, we then have:

$$\kappa \zeta \mathbb{E}_0 \left[\sum_{t=0}^{\infty} \beta^t \widehat{y}_t \right] \approx -\frac{\kappa}{2} \mathbb{E}_0 \left[\sum_{t=0}^{\infty} \beta^t \left\{ \frac{\epsilon(1+\varphi)}{\kappa}\pi_t^2 + [(\sigma+\varphi)(2+\varphi-\sigma) + (1+\varphi)\gamma(1-\gamma)]\widehat{y}_t^2 + \right.\right.$$
$$\left.\left. + (2+\varphi)(1+\varphi)\widehat{\Lambda}_t^2 - 2(1+\varphi)^2[\widehat{y}_t a_t + \widehat{y}_t \widehat{\Lambda}_t - a_t \widehat{\Lambda}_t] \right\} \right] .$$

Expressing in terms of the discounted sum of output:

$$\mathbb{E}_0 \left[\sum_{t=0}^{\infty} \beta^t \widehat{y}_t \right] \approx \frac{1}{2\zeta} \mathbb{E}_0 \left[\sum_{t=0}^{\infty} \beta^t \left\{ -\frac{\epsilon(1+\varphi)}{\kappa}\pi_t^2 - [(\sigma+\varphi)(2+\varphi-\sigma) + (1+\varphi)\gamma(1-\gamma)]\widehat{y}_t^2 - \right.\right.$$
$$\left.\left. - (2+\varphi)(1+\varphi)\widehat{\Lambda}_t^2 + 2(1+\varphi)^2[\widehat{y}_t a_t + \widehat{y}_t \widehat{\Lambda}_t - a_t \widehat{\Lambda}_t] \right\} \right] .$$

Having expressed the linear terms in the loss function by quadratic terms, we **plug these quadratic terms back** into the loss function that still contains the linear terms (A.34) to arrive at a loss function in terms of second order terms for inflation, output, and emission damages:

$$\mathcal{L} \approx \frac{1}{2} \mathbb{E}_0 \left[\sum_{t=0}^{\infty} \beta^t \left\{ \Omega_\pi \pi_t^2 + \Omega_y (\widehat{y}_t)^2 + \Omega_\Lambda \widehat{\Lambda}_t^2 - 2\Omega_{ya} \widehat{y}_t a_t - 2\Omega_{y\Lambda} \widehat{y}_t \widehat{\Lambda}_t - 2\Omega_{a\Lambda} \widehat{\Lambda}_t a_t \right\} \right] , \quad \text{(A.37)}$$

with auxiliary parameters

$$
\begin{aligned}
\Omega_\pi &= \frac{\epsilon}{\kappa}\left[\frac{1+\Phi}{1+\gamma} - \Phi\frac{1+\varphi}{\zeta}\right] \\
\Omega_y &= -1+\sigma+\frac{1+\Phi}{1+\gamma}[1+\varphi+\gamma(1-\gamma)] - \Phi\frac{(\sigma+\varphi)(2+\varphi-\sigma)+(1+\varphi)\gamma(1-\gamma)}{\zeta} \\
&= \gamma(1-\gamma)\left[\frac{1+\Phi}{1+\gamma} - \Phi\frac{1+\varphi}{\zeta}\right] - 1+\sigma+\frac{1+\Phi}{1+\gamma}(1+\varphi) - \frac{\Phi}{\zeta}[(1+\varphi)-(1-\sigma)][1+\varphi+1-\sigma] \\
&= \gamma(1-\gamma)\left[\frac{1+\Phi}{1+\gamma} - \Phi\frac{1+\varphi}{\zeta}\right] - 1+\sigma+\frac{1+\Phi}{1+\gamma}(1+\varphi) - \frac{\Phi}{\zeta}[(1+\varphi)^2-(1-\sigma)^2] \\
&= [\gamma(1-\gamma)+1+\varphi]\left[\frac{1+\Phi}{1+\gamma} - \Phi\frac{1+\varphi}{\zeta}\right] - (1-\sigma)\left(1-\frac{\Phi}{\zeta}(1-\sigma)\right) \\
&= [\gamma(1-\gamma)+1+\varphi]\left[\frac{1+\Phi}{1+\gamma} - \Phi\frac{1+\varphi}{\zeta}\right] - (1-\sigma)\left(1-\frac{\Phi}{\zeta}(1+\varphi+\gamma(1+\varphi)-\zeta)\right) \\
&= [\gamma(1-\gamma)+1+\varphi-(1-\sigma)(1+\gamma)]\left[\frac{1+\Phi}{1+\gamma} - \Phi\frac{1+\varphi}{\zeta}\right] \\
\Omega_\Lambda &= (2+\varphi)\left[\frac{1+\Phi}{1+\gamma} - \Phi\frac{1+\varphi}{\zeta}\right] \\
\Omega_{ya} &= (1+\varphi)\left[\frac{1+\Phi}{1+\gamma} - \Phi\frac{1+\varphi}{\zeta}\right] \\
\Omega_{y\Lambda} &= (1+\varphi)\left[\frac{1+\Phi}{1+\gamma} - \Phi\frac{1+\varphi}{\zeta}\right] \\
\Omega_{a\Lambda} &= -(1+\varphi)\left[\frac{1+\Phi}{1+\gamma} - \Phi\frac{1+\varphi}{\zeta}\right]
\end{aligned}
$$

Eliminating the linear terms in (A.37) using $\widehat{\Lambda}_t = -\gamma\widehat{y}_t - \frac{1}{2}\gamma(1-\gamma)\widehat{y}_t^2 - \frac{1}{2}\widehat{\Lambda}_t^2$, we get:

$$
\mathcal{L} \approx \frac{1}{2}\mathbb{E}_0\left[\sum_{t=0}^{\infty}\beta^t\left\{\Omega_\pi\pi_t^2 + \widetilde{\Omega}_y\widehat{y}_t^2 - 2\widetilde{\Omega}_{ya}\widehat{y}_t a_t\right\}\right], \tag{A.38}
$$

with two composite auxiliary parameters associated with the squared output response $\widehat{y}_t^2$ and the interaction term $\widehat{y}_t a_t$:

$$
\begin{aligned}
\widetilde{\Omega}_y &= \Omega_y + \gamma^2\cdot\Omega_\Lambda + 2\gamma\Omega_{y\Lambda} = \left[\frac{1+\Phi}{1+\gamma} - \Phi\frac{1+\varphi}{\zeta}\right][\zeta(1+\gamma)+\gamma], \\
\widetilde{\Omega}_{ya} &= \Omega_{ya} - \gamma\Omega_{a\Lambda} = \left[\frac{1+\Phi}{1+\gamma} - \Phi\frac{1+\varphi}{\zeta}\right](1+\varphi)(1+\gamma).
\end{aligned}
$$

To ease interpretation, we substitute the definition of the natural output gap

$\widehat{y}_t = x_t^n + \widehat{y}_t^n = x_t^n + \frac{1+\varphi}{\zeta} a_t$:

$$\mathcal{L} \approx \frac{1}{2}\mathbb{E}_0\left[\sum_{t=0}^{\infty}\beta^t\left\{\Omega_\pi \pi_t^2 + \Omega_x (x_t^n)^2 - 2\Omega_{xa} x_t^n a_t\right\}\right]$$
$$= \frac{1}{2}\mathbb{E}_0\left[\sum_{t=0}^{\infty}\beta^t\left\{\Omega_\pi \pi_t^2 + \Omega_x\left((x_t^n)^2 - 2\frac{\Omega_{xa}}{\Omega_x} x_t^n a_t\right)\right\}\right], \tag{A.39}$$

with auxiliary parameters

$$\Omega_x = \widetilde{\Omega}_y \quad \text{and} \quad \Omega_{xa} = \widetilde{\Omega}_{ya} - \frac{1+\varphi}{\zeta}\widetilde{\Omega}_y \; .$$

We can exploit that the expression $\left[\frac{1+\Phi}{1+\gamma} - \Phi\frac{1+\varphi}{\zeta}\right]$ cancels out when normalizing the weight on inflation in the loss function to one. Adding and subtracting $\frac{\Omega_{xa}^2}{\Omega_x^2}$ delivers an expression for the loss function in terms of squared inflation and natural output deviations from a target level x_t^*:

$$\mathcal{L} \approx \frac{1}{2}\mathbb{E}_0\left[\sum_{t=0}^{\infty}\beta^t\left\{\pi_t^2 + \omega_x\left(x_t - x_t^*\right)^2\right\}\right], \tag{A.40}$$

with $\omega_x = \frac{\Omega_x}{\Omega_\pi} = \frac{\kappa}{\epsilon}(\zeta(1+\gamma)+\gamma)$ and $x_t^* = \frac{\Omega_{xa}}{\Omega_x} a_t = -\frac{1+\varphi}{\zeta}\frac{\gamma}{\zeta(1+\gamma)+\gamma} a_t$. □

B Additional Numerical Results: Efficient Long Run Tax

In the main text, we have focused on the case where the efficient carbon tax is set to zero, i.e. there is a distorted deterministic steady state. In this section, we show numerically that our characterization of monetary policy does not depend on this assumption. Table B.1 displays key macro moments under Ramsey optimal policy for different values of the Calvo parameter θ in the textbook model (left two columns), the simplified model where emission damages depend on the emission flow (middle columns) and the baseline model where damages depend on the persistent emission stock (right two columns).

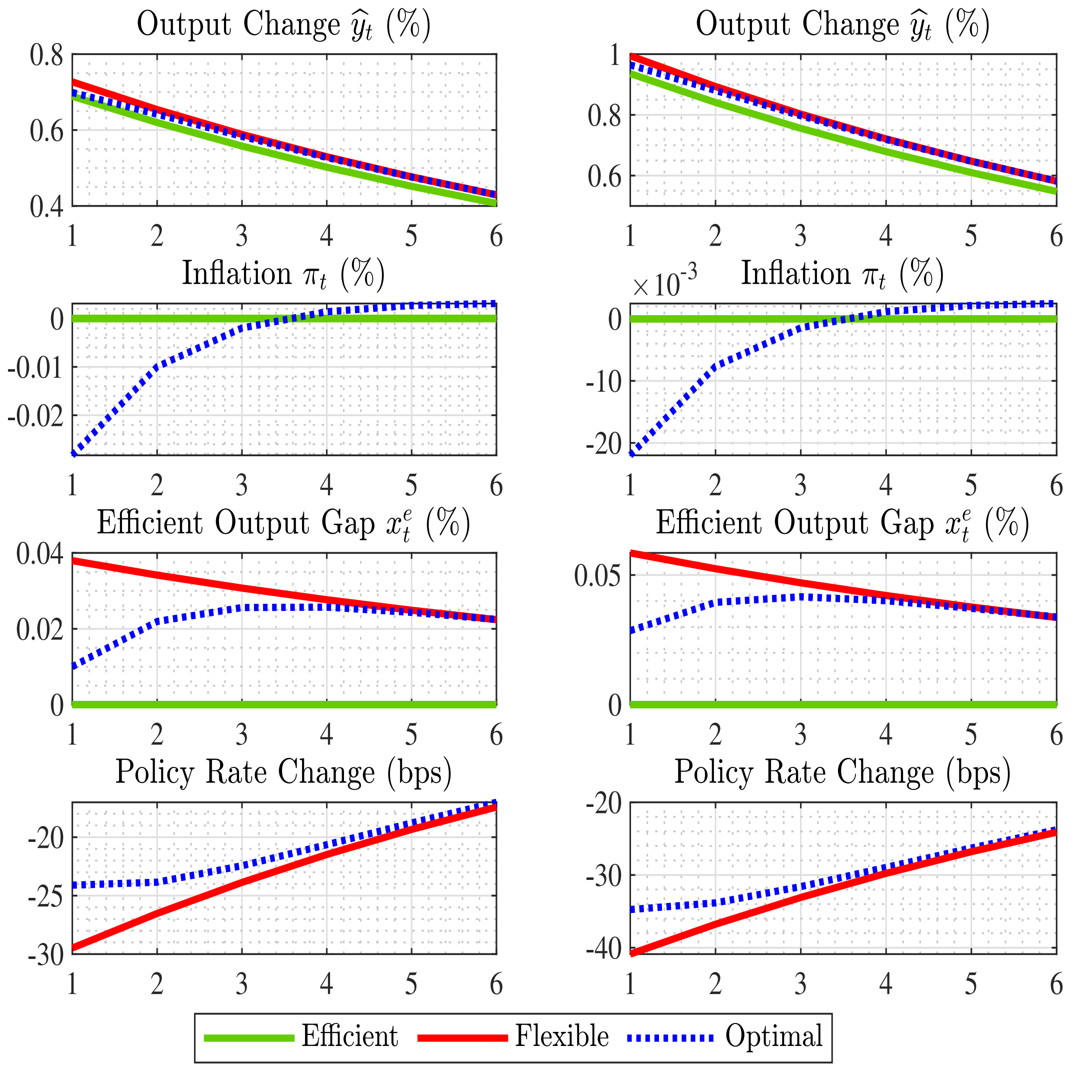
Table B.1: Macro and Welfare Effects with Efficient Steady State Tax

	$\gamma = 0$		$\gamma = 0.37$			
			$\delta = 1$		$\delta = 0.0035$	
Calvo Parameter θ	0.6	0.8	0.6	0.8	0.6	0.8
Volatility						
Output Dev. $\widehat{y}_t$ (%)	2.29	2.29	1.67	1.65	2.27	2.25
Eff. Output Gap x_t^e (%)	0	0	0.12	0.11	0.13	0.12
Nat. Output Gap x_t^n (%)	0	0	0.01	0.03	0.01	0.03
Inflation (%)	0	0	0.05	0.03	0.04	0.03
Policy Rate (bps)	93	93	65	63	91	88
Welfare Effect (CE, %)						
Gain of Optimal MP	0.03	0.08	0.007	0.024	0.011	0.049

Notes: All moments are computed under optimal monetary policy in the presence of a constant carbon tax that renders the deterministic steady state efficient. Inflation volatility is annualized an expressed in percentage points, the policy rate is annualized and expressed in basis points. We express the welfare gain of optimal monetary policy in consumption equivalents $gain^{CE,opt} \equiv \exp\{(1-\beta)(V^{opt} - V^{base})\} - 1$, where V^{base} refers to welfare (29) in an economy where the central bank follows a simple monetary policy rule (15) with inflation coefficient $\phi = 1.5$.

The right panel of Figure B.1 shows impulse response to a TFP shock for the case of persistent emissions.

Figure B.1: IRF to TFP-Shock: Optimal Monetary Policy



Notes: Impulse response to a positive one standard deviation shock to TFP, using a second order approximation around the deterministic steady state. The left column refers to the three-equation model where damages depend on the flow of emissions while the right column refers to the four-equation model where damages depend on a persistent stock of emissions.

C Central Bank Loss Function

In this section, we derive the central bank loss function that does not take into account that output expansions affect economic damages from emissions. In this case, approximating household utility and the NKPC up to second order and combining it with the production function and market clearing condition (Benigno and Woodford 2005) yields a loss function that inherits the inefficiency of the flexible price equilibrium. Consequently, this loss function prescribes to close natural output gap and inflation gap to zero in all states, which is feasible in our model. Notably, this loss function takes the presence of climate change into account in so far as it affects the natural rate of interest, i.e. it is *climate conscious* (Nakov and Thomas 2023). However, such a loss function is not consistent with an utilitarian welfare criterion.

The **second order approximation** of the welfare objective, i.e. households period utility function U_t, remains unchanged:

$$\frac{U_t - U}{c^{1-\sigma}} \approx \widehat{c}_t + \frac{\widehat{c}_t^2}{2} - \frac{\sigma}{2}\widehat{c}_t^2 - (1-\tau^c)\left[\widehat{n}_t + \frac{\widehat{n}_t^2}{2} + \frac{\varphi}{2}\widehat{n}_t^2\right].$$

As in the main text, in order to express the loss function in terms of the natural output gap x_t^n and inflation π_t, we make use of the market clearing condition $\widehat{c}_t = \widehat{y}_t$. Different to before, we directly exploit (A.23) to re-write the production function as $\widehat{n}_t = \widehat{y}_t + \widehat{\Delta}_t - a_t - \widehat{\Lambda}_t = (1+\gamma)\widehat{y}_t + \widehat{\Delta}_t - a_t$:

$$\frac{U_t - U}{c^{1-\sigma}} \approx \widehat{y}_t + \frac{1-\sigma}{2}\widehat{y}_t^2 - (1-\tau)(1+\gamma)\left[\widehat{y}_t + \frac{\widehat{\Delta}_t}{1+\gamma} - \frac{a_t}{1+\gamma} + \frac{1+\varphi}{2(1+\gamma)}\left((1+\gamma)\widehat{y}_t + \widehat{\Delta}_t - a_t\right)^2\right],$$

such that $\widehat{\Lambda}_t$ does not enter the objective function. Eliminating all terms independent of policy and of order higher than two, we then obtain:

$$\begin{aligned}\frac{U_t - U}{c^{1-\sigma}} &\approx \widehat{y}_t + \frac{1-\sigma}{2}\widehat{y}_t^2 - (1+\Phi)\left[\widehat{y}_t + \widehat{\Delta}_t\frac{1}{1+\gamma} + \frac{1+\varphi}{2}(1+\gamma)\widehat{y}_t^2 - (1+\varphi)y_t a_t)^2\right] + t.i.p. \\ &\approx -\Phi\widehat{y}_t - \frac{\widehat{y}_t^2}{2}\left[-1+\sigma+(1+\Phi)(1+\varphi)(1+\gamma)\right] - \frac{1+\Phi}{1+\gamma}\widehat{\Delta}_t + (1+\Phi)(1+\varphi)\widehat{y}_t a_t + t.i.p. \\ &\approx -\Phi\widehat{y}_t - \frac{\widehat{y}_t^2}{2}\left[\zeta(1+\Phi)+\Phi(1-\sigma)\right] - \frac{1+\Phi}{1+\gamma}\widehat{\Delta}_t + (1+\Phi)(1+\varphi)\widehat{y}_t a_t + t.i.p.\end{aligned}$$

Using the definition of the natural output level (10) and of the natural output gap $\widehat{y}_t = x_t^n + \widehat{y}_t^n = x_t^n + \frac{1+\varphi}{\zeta}a_t$ and eliminating all terms independent of policy, we arrive at:

$$\begin{aligned}\frac{U_t - U}{c^{1-\sigma}} &\approx -\Phi x_t^n - \frac{\zeta(1+\Phi)+\Phi(1-\sigma)}{2}\left((x_t^n)^2 + 2\frac{1+\varphi}{\zeta}x_t^n a_t\right) - \frac{1+\Phi}{1+\gamma}\widehat{\Delta}_t + \\ &\quad + (1+\Phi)(1+\varphi)x_t^n a_t + t.i.p. \\ &\approx -\Phi x_t^n - \frac{1}{2}\left\{\zeta(1+\Phi)+\Phi(1-\sigma)\right\}(x_t^n)^2 - \frac{1+\Phi}{1+\gamma\widetilde{\Phi}}\widehat{\Delta}_t - \frac{\Phi}{\zeta}(1-\sigma)(1+\varphi)x_t^n a_t + t.i.p.\end{aligned}$$

We are then ready to **evaluate the loss function**:

$$\mathcal{L} \equiv -\mathcal{W} \approx \mathbb{E}_0\left[\sum_{t=0}^{\infty}\beta^t\left\{\frac{1}{2}\Big(\zeta(1+\Phi)+\Phi(1-\sigma)\Big)(x_t^n)^2+\frac{1+\Phi}{1+\gamma}\widehat{\Delta}_t+\right.\right.$$
$$\left.\left.+\Phi\frac{(1+\varphi)}{\zeta}(1-\sigma)x_t^n a_t+\Phi x_t^n\right\}\right] .$$

The discounted sum of log price dispersions is given by $\sum_{t=0}^{\infty}\beta^t\widehat{\Delta}_t \approx \frac{\epsilon}{2\kappa}\sum_{t=0}^{\infty}\beta^t\pi_t^2$, with the auxiliary parameter $\kappa = \frac{(1-\theta)(1-\theta\beta)}{\theta}$ governing the slope of the NKPC. Therefore, the loss function is given by

$$\mathcal{L} \approx \mathbb{E}_0\left[\sum_{t=0}^{\infty}\beta^t\left\{\frac{1}{2}\Big(\zeta(1+\Phi)+\Phi(1-\sigma)\Big)(x_t^n)^2+\frac{1+\Phi}{1+\gamma}\frac{\epsilon}{2\kappa}\widehat{\Delta}_t\right.\right.$$
$$\left.\left.+\Phi\frac{(1+\varphi)}{\zeta}(1-\sigma)x_t^n a_t+\Phi x_t^n\right\}\right] . \tag{C.1}$$

Eliminating Linear Terms. There is only one linear term Φx_t^n in this expression and we exploit that the same (extended) NKPC (A.35), which can be iterated forward to:

$$H_0 = \kappa\mathbb{E}_0\left[\sum_{t=0}^{\infty}\beta^t\left\{\widehat{\xi}_{1t}-\widehat{\xi}_{2t}+\frac{1}{2}(\widehat{\xi}_{1t}^2-\widehat{\xi}_{2t}^2)\right\}\right]+\frac{\epsilon}{2}\sum_{t=0}^{\infty}\beta^t\pi_t^2 . \tag{C.2}$$

Now, using market clearing (A.30) and the re-written production function $\widehat{n}_t = (1+\gamma)\widehat{y}_t + \widehat{\Delta}_t - a_t$, we can write the Calvo terms without taking the terms related to $\widehat{\Lambda}_t$ into account:

$$\begin{aligned}\widehat{\xi}_{1t} =& \widehat{mc}_t+\widehat{y}_t-\sigma\widehat{c}_t = \widehat{w}_t - a_t+\gamma\widehat{y}_t+\widehat{y}_t-\sigma\widehat{y}_t = \varphi\widehat{n}_t - a_t + (1+\gamma)\widehat{y}_t\\ =&\varphi(\widehat{\Delta}_t+\widehat{y}_t-a_t+\gamma\widehat{y}_t)-a_t+(1+\gamma)\widehat{y}_t = (1+\zeta-\sigma)\widehat{y}_t-(1+\varphi)a_t+\varphi\widehat{\Delta}_t ,\end{aligned}$$

Hence, the difference between the Calvo terms reduces to:

$$\widehat{\xi}_{1t}-\widehat{\xi}_{2t} = (1+\zeta)\widehat{y}_t-(1+\varphi)a_t+\varphi\widehat{\Delta}_t-\widehat{y}_t = \zeta\widehat{y}_t-(1+\varphi)a_t+\varphi\widehat{\Delta}_t \approx \zeta\widehat{y}_t+\varphi\widehat{\Delta}_t .$$

The difference between the squared Calvo terms, ignoring terms of order higher than two and terms irrelevant for policy, simplifies to:

$$\begin{aligned}\widehat{\xi}_{1t}^2-\widehat{\xi}_{2t}^2 =& [(1+\zeta-\sigma)\widehat{y}_t-(1+\varphi)a_t+\varphi\widehat{\Delta}_t]^2-(1-\sigma)^2\widehat{y}_t^2\\ =&[(1+\zeta-\sigma)^2-1]\widehat{y}_t^2-2(1+\zeta-\sigma)(1+\varphi)\widehat{y}_t a_t\\ =&\zeta(\zeta+2-2\sigma)\widehat{y}_t^2-2(1+\zeta-\sigma)(1+\varphi)\widehat{y}_t a_t .\end{aligned}$$

Hence, we can then rewrite the discounted sum H_0 as

$$H_0 \approx \kappa \mathbb{E}_0 \left[\sum_{t=0}^{\infty} \beta^t \left\{ \zeta y_t + \varphi \Delta_t + \frac{1}{2}\zeta(\zeta + 2 - 2\sigma)\widehat{y}_t^2 - (1 + \zeta - \sigma)(1 + \varphi)\widehat{y}_t a_t + \frac{\epsilon}{2\kappa}\pi_t^2 \right\} \right. . \tag{C.3}$$

Since H_0 is given and the price dispersion terms can be expressed in terms of inflation $\sum \beta^t \Delta_t = \frac{\epsilon}{2\kappa} \sum \beta^t \pi_t^2$, we then can rearrange (C.3) for the discounted sum of output deviations:

$$\mathbb{E}_0 \left[\sum_{t=0}^{\infty} \beta^t \widehat{y}_t \right] \approx \frac{1}{2\zeta} \mathbb{E}_0 \left[\sum_{t=0}^{\infty} \beta^t \left\{ -\frac{\epsilon(1+\varphi)}{\kappa}\pi_t^2 - \zeta(\zeta + 2 - 2\sigma)\widehat{y}_t^2 + 2(1 + \zeta - \sigma)(1 + \varphi)\widehat{y}_t a_t \right\} \right. .$$

Rewriting in terms of the natural output gap $\widehat{y}_t = x_t^n + \widehat{y}_t^n = x_t^n + \frac{1+\varphi}{\zeta} a_t$, and ignoring terms of order higher than two and irrelevant for policy, we get:

$$\begin{aligned} \mathbb{E}_0 \left[\sum_{t=0}^{\infty} \beta^t x_t^n \right] \approx \frac{\mathbb{E}_0}{2\zeta} \left[\sum_{t=0}^{\infty} \beta^t \left\{ - \frac{\epsilon(1+\varphi)}{\kappa}\pi_t^2 - \zeta(\zeta + 2 - 2\sigma)\Big((x_t^n)^2 + 2\frac{1+\varphi}{\zeta}x_t^n a_t\Big) \right. \right. \\ \left. \left. + 2(1 + \zeta - \sigma)(1 + \varphi)x_t^n a_t \right\} \right] , \end{aligned}$$

which implies that we can express the present value of natural output gaps as

$$\mathbb{E}_0 \left[\sum_{t=0}^{\infty} \beta^t x_t^n \right] \approx \mathbb{E}_0 \left[\sum_{t=0}^{\infty} \beta^t \left(\frac{1}{2}X_1\pi_t^2 + \frac{1}{2}X_2(x_t^n)^2 + X_3 x_t^n a_t \right) \right] , \tag{C.4}$$

where the auxiliary terms are defined as

$$X_1 = -\frac{\epsilon}{\kappa}\frac{1+\varphi}{\zeta} \; , \; X_2 = -(\zeta + 2 - 2\sigma), \; X_3 = -(1+\varphi)\frac{1-\sigma}{\zeta} \; .$$

Having expressed the linear terms in the loss function by quadratic terms, we **plug**

these quadratic terms back into (C.1):

$$\begin{aligned}
\mathcal{L} \approx \mathbb{E}_0\Bigg[\sum_{t=0}^{\infty}\beta^t\Bigg\{&\frac{1}{2}\Big(\zeta + \Phi(\zeta+1-\sigma)\Big)(x_t^n)^2 + \frac{1+\Phi}{1+\gamma}\frac{\epsilon}{2\kappa}\pi_t^2 \\
&+ \Phi\frac{1+\varphi}{\zeta}(1-\sigma)x_t^n a_t + \Phi x_t^n\Bigg\}\Bigg] \\
\approx \mathbb{E}_0\Bigg[\sum_{t=0}^{\infty}\beta^t\Bigg\{&\frac{1}{2}\Big(\zeta + \Phi(\zeta+1-\sigma) + \Phi X_2\Big)(x_t^n)^2 + \frac{1}{2}\Big(\frac{1+\Phi}{1+\gamma}\frac{\epsilon}{\kappa} + \Phi X_1\Big)\pi_t^2 \\
&+ \Big(\frac{1+\varphi}{\zeta}\Big[\Phi(1-\sigma)\Big] + \Phi X_3\Big)\, x_t^n a_t\Bigg\}\Bigg] \\
\approx \mathbb{E}_0\Bigg[\sum_{t=0}^{\infty}\beta^t\Bigg\{&\frac{1}{2}\frac{\epsilon}{\kappa}\frac{\zeta - \Phi(1-\sigma)}{\zeta(1+\gamma)}\pi_t^2 + \frac{1}{2}\Big(\zeta - \Phi(1-\sigma)\Big)(x_t^n)^2\Bigg\}\Bigg] \\
\approx \frac{1}{2}\mathbb{E}_0\Bigg[\sum_{t=0}^{\infty}\beta^t&\Bigg\{\Omega_\pi \pi_t^2 + \Omega_x (x_t^n)^2\Bigg\}\Bigg] \,, \qquad \text{(C.5)}
\end{aligned}$$

with auxiliary parameters

$$\Omega_\pi = \frac{\epsilon}{\kappa}\frac{\zeta - \Phi(1-\sigma)}{\zeta(1+\gamma)} \,, \; \Omega_x = \zeta - \Phi(1-\sigma) \,.$$

As a last step, we get:

$$\mathcal{L} \approx \frac{1}{2}\mathbb{E}_0\Bigg[\sum_{t=0}^{\infty}\beta^t\Bigg\{\pi_t^2 + \omega_x\,(x_t^n)^2\Bigg\}\Bigg] \,, \qquad \text{(C.6)}$$

where $\omega_x = \frac{\Omega_x}{\Omega_\pi} = \frac{\kappa}{\epsilon}\zeta(1+\gamma)$. □

Different to the loss function in the main text, (C.6) does not include a target level for the natural output gap. While the weight on output stabilization is still positively related to the externality, divine coincidence still holds: a central bank that chooses not to internalize the effect of output expansions on economic damages associated with emission externalities can and should perfectly close inflation and natural output gap.