Stationary Rational Bubbles in Non-Linear Business Cycle Models

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Main result:

non-linear DSGE models have more stationary equilibria than you think!

This paper shows: standard NON-LINEAR DSGE models have MULTIPLE stationary equilibria, even when the linearized versions of these models have unique solution

⇒ In non-linear model: stationary fluctuations WITHOUT shocks to TFP, preferences, policy

$\Rightarrow$ Blanchard & Kahn (1980): conditions for existence of unique **stable** solution of **linear(ized)** models are IRRELEVANT for non-linear models

$\Rightarrow$ Sunspot equilibria in non-linear models studied here look like 'BUBBLES':

- **economy may temporarily diverge from steady state;**
- **with exogenous probability economy later reverts to steady state**

BOOM-BUST CYCLE:

- **consistent with rational expectations**
- **'rational bubbles' are stationary**

Similarities and important differences with rational bubbles in linear models (Blanchard, 1979)

- **Like Blanchard (1979) I focus on <u>models whose linearized versions have unique non-explosive equilibrium</u>**

- **Key difference: bubbles in <u>non-linear</u> models are <u>STATIONARY</u>**

- **Blanchard bubbles (linear models): *expected* trajectories <u>explode</u> to ± ∞**

Consider non-linear model with just 1 non-predetermined variable (no exogenous driver)

$$E_t G(Y_{t+1}, Y_t) = 0$$

Linearization (around steady state):

$$E_t y_{t+1} = \lambda \cdot y_t, \quad y_t \equiv Y_t - Y^{SS}$$

Linearized model has unique non-explosive solution iff $|\lambda| > 1$. Unique solution is: $y_t = 0$

(Blanchard & Kahn (1980), Prop. 1)

$E_t y_{t+1} = \lambda \cdot y_t, \quad \lambda > 1;$ $\quad y_t$: scalar jump variable

Unique stable solution: $y_t = 0$

Blanchard (1979)

Bubble: $y_{t+1} = (\lambda/(1-\pi)) \cdot y_t$ with probability $1-\pi$

$y_{t+1} = 0$ with probability π

$\lim_{s\to\infty} E_t y_{t+s} = \pm\infty$ if $y_t \neq 0$

expected path of bubble diverges to ±∞

Expected path of bubbles in <u>non-linear</u> DSGE described here do NOT diverge to ±∞

● Explosive (expected) trajectories are problematic:
► accuracy of linear model approximations breaks down far from point of approximation; non-negativity & technological feasibility constraints may be violated

Example: with decreasing returns to capital, explosive trajectory of capital & output is INFEASIBLE

⇒ LINEAR APPROXIMATION UNSUITABLE FOR ANALYZING RATIONAL BUBBLES

- **By contrast: <u>non-linear</u> analysis here takes non-negativity constraints, decreasing returns & risk aversion into account**

- **Decreasing returns & risk aversion generate stabilizing forces that prevent explosive trajectories**

- **Stationary rational bubbles in <u>non-linear</u> models are generally <u>one-sided</u> (capital over-accumulation, but no under-accumulation)**
[By contrast: Blanchard bubbles in linear models can be positive or negative]

- **Rational bubbles in <u>non-linear</u> model can induce fluctuations that are <u>close to deterministic steady state most of the time</u>**

$\Rightarrow$ unconditional mean of endogenous variables close to deterministic steady state

- **Non-linear DSGE models driven just by stationary bubbles can generate <u>persistent fluctuations of real activity</u> & <u>capture key business cycle stylized facts</u>**

Note: Can construct DSGE models whose linearized versions have stable sunspots:

$E_t y_{t+1} = \lambda \cdot y_t$ **need** $|\lambda| \leq 1$. $\Rightarrow y_{t+1} = \lambda \cdot y_t + \varepsilon_{t+1}$ is stationary solution for <u>any</u> $\{\varepsilon_{t+1}\}$ with $E_t \varepsilon_{t+1} = 0$

Needed ingredients:

● Increasing returns, externalities (e.g., Schmitt-Grohé (1997), Benhabib and Farmer (1999))
● Financial frictions (e.g., Martin and Ventura (2018))
● Overlapping generations (e.g., Woodford (1986), Galí (2018))

Specific assumptions & calibrations that deliver $|\lambda|<1$ can be debatable & fragile (e.g. in standard OLG model: need r≤g)

<u>By contrast, paper here argues that very standard DSGE models with</u> $|\lambda|>1$ ***<u>can deliver stationary sunspot equilibria, if non-linearities are considered.</u>***

Related contributions

- **Bacchetta, van Wincoop & Tille "Self-fulfilling Risk Panics" (AER 2012): stylized asset pricing model whose linearized version has unique solution, but non-linear model has multiple equilibria iff sunspot shocks are HETEROSKEDASTIC.**

My paper highlights importance of heteroskedasticity for bubbles in non-linear DSGE business cycle model.

- **Holden (2016ab) shows that multiple equilibria emerge when occasionally binding constraints (e.g. ZLB) are integrated into otherwise standard linear model.**

- **By contrast: my analysis considers FULLY <u>non-linear</u> models.**
- **All model equations are non-linear**
- **All relevant non-negativity constraints are imposed.**
- **Model solutions here are globally accurate.**
- **Multiple equilibria here have "bubbly" dynamics (different from Holden, 2016ab)**

Basic intuition I:

Consider non-linear model with just 1 non-predetermined variable (no exogenous driver)

$$E_t G(Y_{t+1}, Y_t) = 0$$

Linearization (around steady state):

$$E_t y_{t+1} = \lambda \cdot y_t, \quad y_t \equiv Y_t - Y^{SS}$$

Linearized model has unique non-explosive solution iff $|\lambda| > 1$. Unique solution is: $y_t = 0$ (Blanchard & Kahn (1980), Prop. 1)

• RESULT

Even when $|\lambda|>1$, the **non-linear** model can have stationary sunspot equilibrium

• IDEA

$E_t G(Y_{t+1}, Y_t)=0 \Leftrightarrow G(Y_{t+1}, Y_t)=\varepsilon_{t+1}$ with $E_t \varepsilon_{t+1}=0$

$\Rightarrow Y_{t+1}=\Lambda(Y_t, \varepsilon_{t+1})$. ε_{t+1}: "sunspot shock"

Even if $|\Lambda_Y|>1$, there may exist process $\{\varepsilon_{t+1}\}$ with $E_t \varepsilon_{t+1}=0$ such that $\{Y_{t+1}\}$ is stationary.

Note: when white noise $\{\varepsilon_{t+1}\}$ is fed into $Y_{t+1}=\Lambda(Y_t, \varepsilon_{t+1})$, then $\{Y_{t+1}\}$ diverges if $|\Lambda_Y|>1$.

Key requirements for stationary solution:

- $Y_{t+1}=\Lambda(Y_t,\varepsilon_{t+1})$ has to be NON-LINEAR in ε_{t+1}
- Distribution of ε_{t+1} has to depend on Y_t

$$Y_{t+1}\cong\Lambda(Y_t,0)+\Lambda_\varepsilon(Y_t,0)\cdot\varepsilon_{t+1}+\tfrac{1}{2}\Lambda_{\varepsilon\varepsilon}(Y_t,0)\cdot(\varepsilon_{t+1})^2$$

$$E_tY_{t+1}\cong\Lambda(Y_t,0)+\tfrac{1}{2}\Lambda_{\varepsilon\varepsilon}(Y_t,0)\cdot E_t(\varepsilon_{t+1})^2$$

Let $E_t(\varepsilon_{t+1})^2=f(Y_t)\geq 0$. If $\Lambda_{\varepsilon\varepsilon}(Y_t,0)\neq 0$ then can set $E_t(\varepsilon_{t+1})^2=f(Y_t)$ such that $|dE_tY_{t+1}/dY_t|<1$:
"MEAN REVERSION"

Example: $\Lambda_Y(Y_t,0)>1$, $\Lambda_{\varepsilon\varepsilon}(Y_t,0)<0$. Then need $f'(Y_t)>0$ for mean reversion: $E_t(\varepsilon_{t+1})^2$ must be <u>increasing</u> in Y_t.

Basic intuition II: RBC model

$C_t + K_{t+1} = Y_t; \quad Y_t = F(K_t), \;\; F' > 0, \;\; F'' < 0$

$\beta\{[E_t u'(C_{t+1})]/u'(C_t)\} \cdot F'(K_{t+1}) = 1$; assume $u''' > 0$ (CRRA)

Sunspot: assume $K_{t+1} \uparrow \Rightarrow C_t \downarrow \;\; u'(C_t) \uparrow, F'(K_{t+1}) \downarrow$

Euler eqn requires: $E_t u'(C_{t+1}) = E_t u'(F(K_{t+1}) - K_{t+2}) \uparrow$

- In <u>deterministic</u> economy: need $C_{t+1} \downarrow$ & $K_{t+2} \uparrow$

K_{t+2} has to rise **<u>more</u>** than K_{t+1} ! $\Rightarrow$ **<u>K diverges</u>**

- With stochastic sunspot: K_{t+2} random.

$u'(C_{t+1})$ is convex in $K_{t+2} \Rightarrow$ if $Var_t(K_{t+2})$ rises,

$E_t u'(C_{t+1}) \uparrow \Rightarrow E_t K_{t+2}$ can rise less than K_{t+1} !

$\Rightarrow$ possibility of **mean reversion**

TRANSVERSALITY CONDITION (TVC)

Standard DSGE usually assume an infinitely-lived representative agent.

Optimality conditions include transversality condition:

$$\lim_{\tau \to \infty} \beta^{\tau} E_t u'(C_{t+\tau}) K_{t+\tau+1} > 0$$

TVC + Euler eqn. + static efficiency condit. $\Rightarrow$ unique equilibrium.

When TVC does not hold: economy is "dynamically inefficient"

THIS PAPER DISREGARDS TVC

- Goal is to establish existence of stationary rational bubbles in non-linear DSGE models
- Explosive bubbles in linear (Blanchard) too violate TVC

JUSTIFICATIONS OF MODELS WITHOUT TVC

Assume that there is no TVC because agents are finitely lived (N periods)

Novel result about OLG economy:
Assume: (I) Complete financial market that allows all generations alive at both dates *t* **and** *t*+1
(II) each generation receives wealth endowment such that consumption by newborns is time-invariant share of aggregate consumption. (Under log-utility: wealth endowment of newborns has to be time-invariant share of total wealth)
THEN
an 'aggregate' Euler equation holds that is identical to the Euler equation of a representative infinitely lived household:

$$\beta E_t\{u'(C_{t+1})/u'(C_t)\}MPK_{t+1} = 1$$

BUT: there is no TVC in the OLG economy!

OLG structure with efficient intergenerational risk sharing:

justification for macro models that lacks a TVC, but whose other equilibrium conditions are identical to those of standard business cycle models (that assume infinitely lived agents)

Other justifications for disregarding TVC

1) Lansing (2010) disregards the TVC in a Lucas-style asset pricing models with bubbles, arguing that "agents are forward-looking but not to the extreme degree implied by the transversality condition"

2) In richer models with heterogeneous agents and distortions: equilibrium is not solution of decision problem of representative agent.

Detection of TVC violations in stochastic economies: virtually impossible, even with very long simulation runs (billions of periods):

States with very low consumption might only occur with extremely small probabilities.

Detailed Example I:
Long-Plosser RBC model with sunspots

$u(C)=\ln(C)$; $C_t+K_{t+1}=Y_t$; $Y_t=F(K_t)\equiv(K_t)^{\alpha}$, $0<\alpha<1$

Euler equation: $\beta E_t\{u'(C_{t+1})/u'(C_t)\}\cdot F'(K_{t+1})=1$

$\Rightarrow \beta E_t\{C_t/C_{t+1}\}\cdot \alpha Y_{t+1}/K_{t+1}=1$

$\Rightarrow \beta E_t\{(Y_t-K_{t+1})/(Y_{t+1}-K_{t+2})\}\cdot \alpha Y_{t+1}/K_{t+1}=1$

$\Rightarrow \alpha\beta\cdot E_t\{(1-K_{t+1}/Y_t)/(1-K_{t+2}/Y_{t+1})\}\cdot Y_t/K_{t+1}=1$

$\Rightarrow \alpha\beta\cdot E_t\{(1-Z_t)/(1-Z_{t+1})\}/Z_t=1$,

$Z_t\equiv K_{t+1}/Y_t$: investment/output ratio

Textbook solution: $Z_t=\alpha\beta$

$$\alpha\beta \cdot E_t\{(1-Z_t)/(1-Z_{t+1})\}/Z_t = 1$$

Linearization around $Z = \alpha\beta$:

$$E_t z_{t+1} = \lambda \cdot z_t, \quad z_t \equiv Z_t - Z; \quad \lambda \equiv 1/(\alpha\beta) > 1.$$

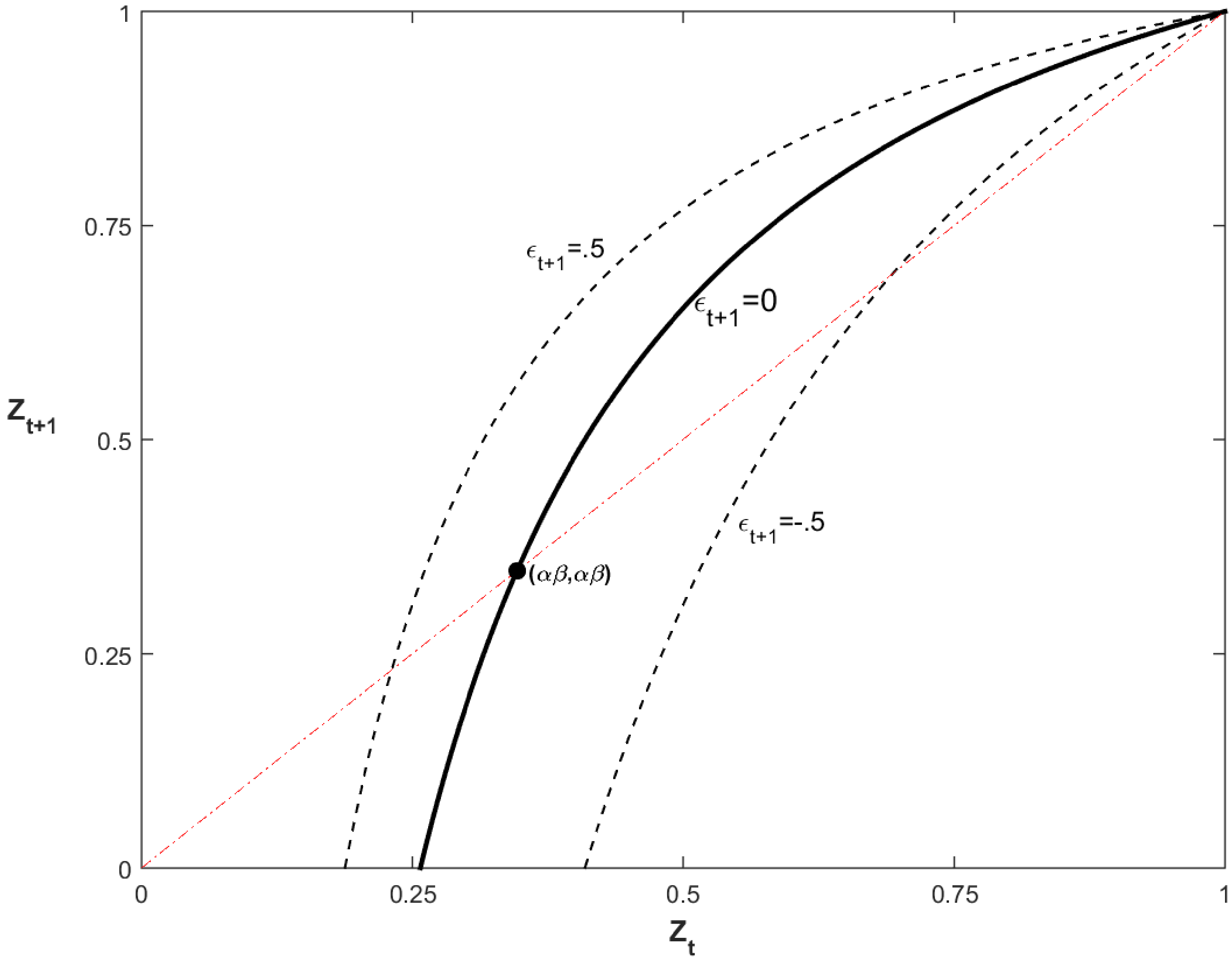
$\Rightarrow z_t = 0$ is unique non-explosive solution of linearized model.

But: **non-linear model has other stationary solutions.**

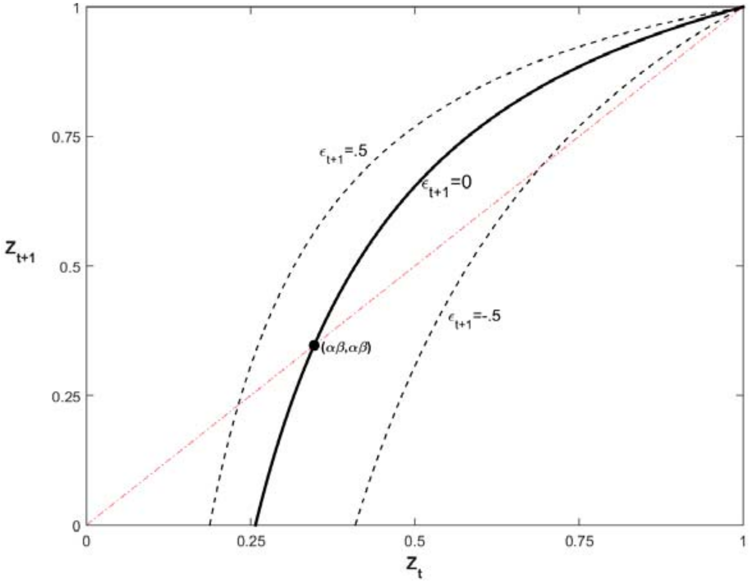
$$\alpha\beta \cdot \{(1-Z_t)/(1-Z_{t+1})\}/Z_t = 1 + \varepsilon_{t+1}, \quad E_t \varepsilon_{t+1} = 0$$

$$\Rightarrow Z_{t+1} = \Lambda(Z_t, \varepsilon_{t+1}) \equiv 1 - \alpha\beta(1/Z_t - 1)/(1+\varepsilon_{t+1}).$$

Z_{t+1} increasing & strictly concave in ε_{t+1}



1
0.75
0.5
0.25
0
0
0.25
0.5
0.75
1
Z_{t+1}
Z_t
ϵ_{t+1}=.5
ϵ_{t+1}=0
ϵ_{t+1}=-.5
($\alpha\beta$,$\alpha\beta$)



$$Z_{t+1}=\Lambda(Z_t,\varepsilon_{t+1}) \equiv 1-\alpha\beta(1/Z_t-1)/(1+\varepsilon_{t+1})$$

- When $Z_t<\alpha\beta$, the model can hit zero-capital corner solution in later periods $\Rightarrow$ restrict attention to solutions with $Z_\tau \in [\alpha\beta,1)\ \forall\, \tau$

- Support of ε_{t+1} has to be bounded below: $\varepsilon_{t+1}\geq -1+[\alpha\beta/(1-\alpha\beta)]\cdot[1/Z_t-1]$ $\Rightarrow$ distribution of ε_{t+1} must depend on Z_t !

- Let ε_{t+1} only takes two values: $-\overline{\varepsilon}_t$ and $\overline{\varepsilon}_t \cdot\pi_t/(1-\pi_t)$ with probabilities π_t and $1-\pi_t$, respectively, $\overline{\varepsilon}_t \in[0,1)$ $\Rightarrow$ Z_{t+1} takes two values:

$$Z^L_{t+1}\equiv\Lambda(Z_t,-\overline{\varepsilon}_t) \text{ \& } Z^H_{t+1}\equiv\Lambda(Z_t,\overline{\varepsilon}_t\pi_t/(1-\pi_t)) \text{ with } Z^L_{t+1}\leq Z^H_{t+1}\leq 1.$$

- Postulate $Z^L_{t+1}=f(Z_t)$, with $\alpha\beta\leq f(Z_t)\leq\Lambda(Z_t,0)$ for $Z_t\in[\alpha\beta,1)$.

Solve $Z^L_{t+1}\equiv\Lambda(Z_t,-\overline{\varepsilon}_t)$ for $\overline{\varepsilon}_t$ & substitute into $Z^H_{t+1}\equiv\Lambda(Z_t,\overline{\varepsilon}_t\pi_t/(1-\pi_t))$

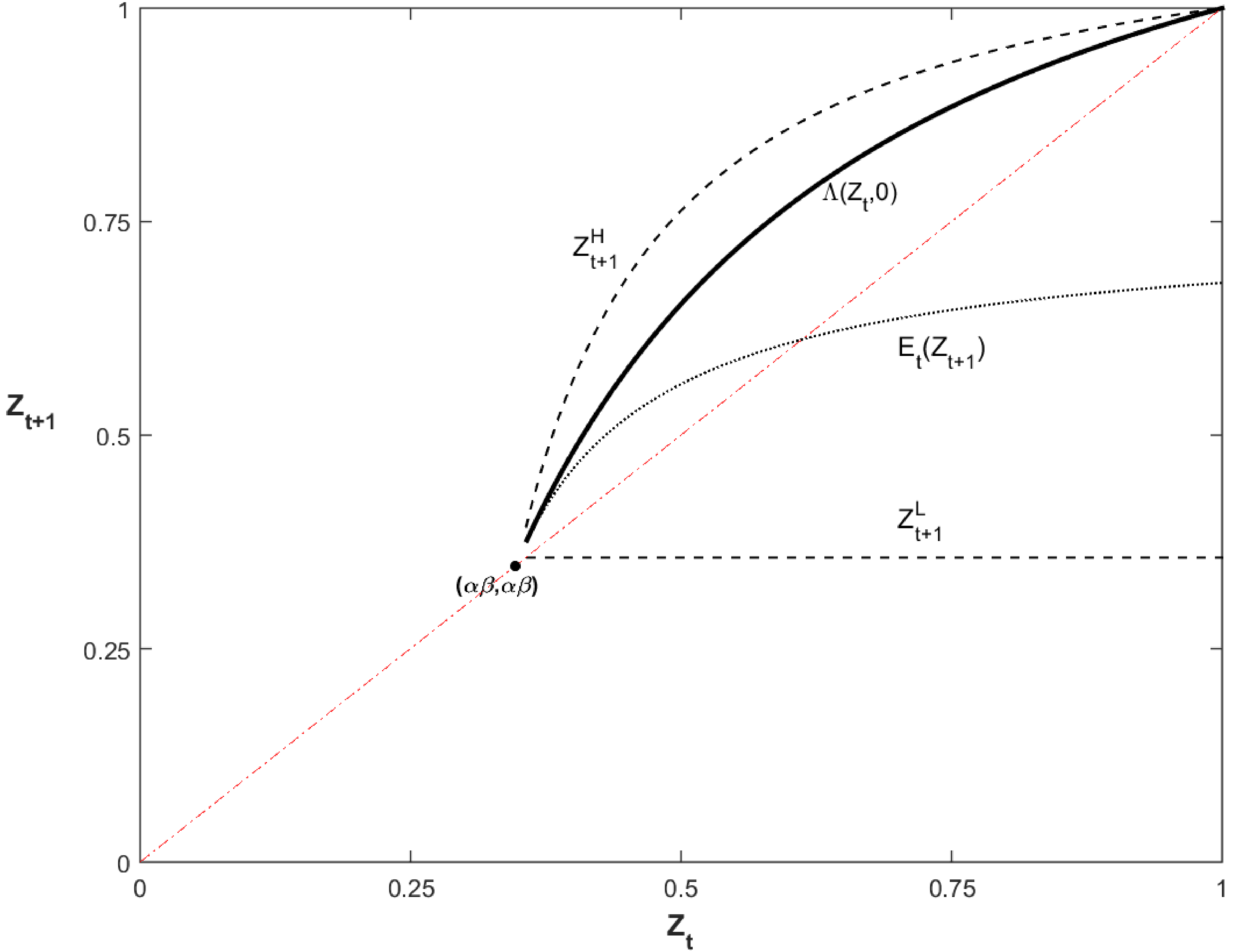
Degrees of freedom in modeling sunspot:

- bust investment/GDP ratio, Z_{t+1}^{L}
- conditional probability of bust, π_t

Specification I: $Z_{t+1}^{L} = \alpha\beta + \Delta,\ \Delta = 0.01,\ \pi = 0.5$

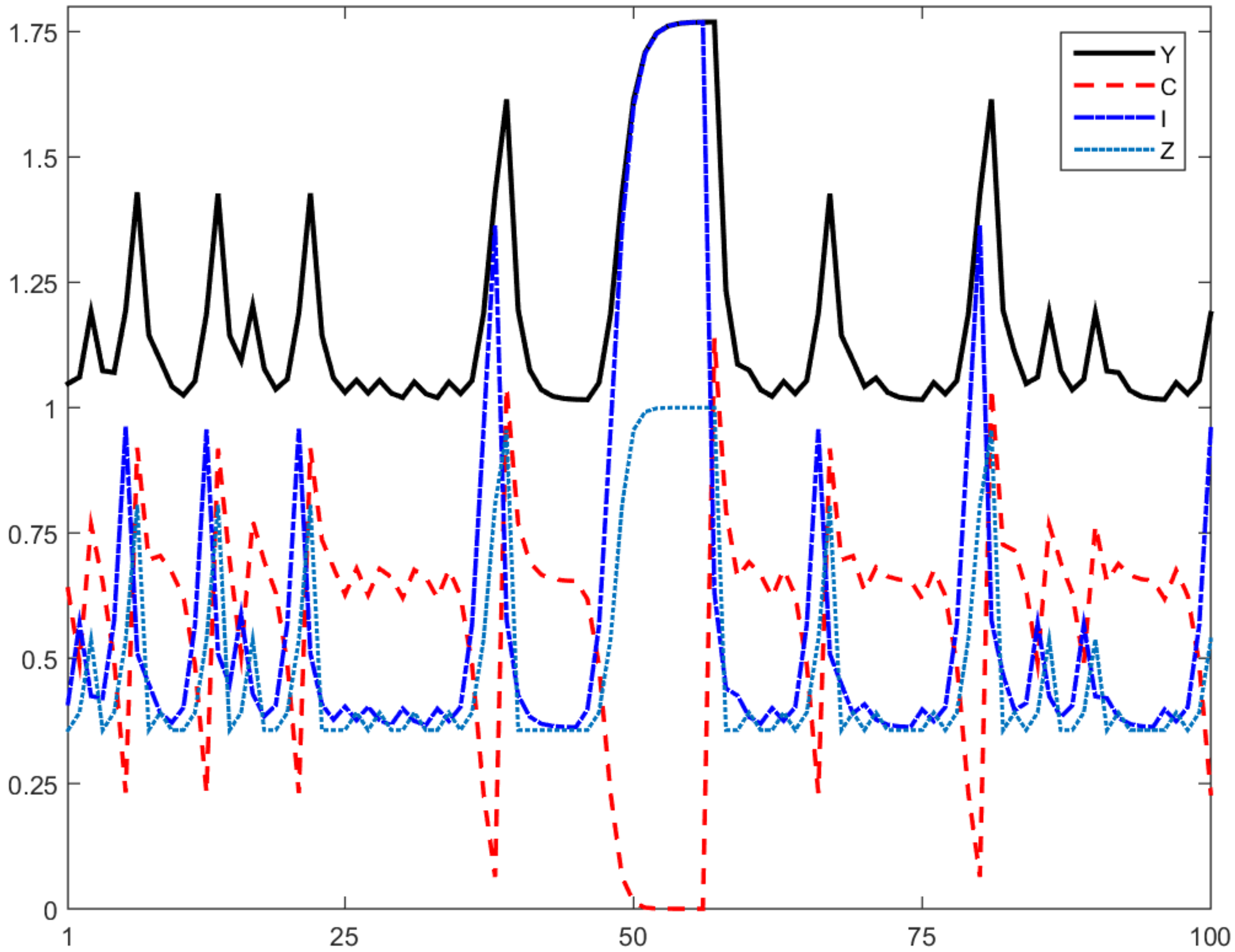
(When $\Delta = 0$, then $Z = \alpha\beta = 0.346$ is absorbing state; thus set $\Delta > 0$)

$Z^L_{t+1} = \alpha\beta + \Delta$, $\Delta = 0.01$, $\pi = 0.5$

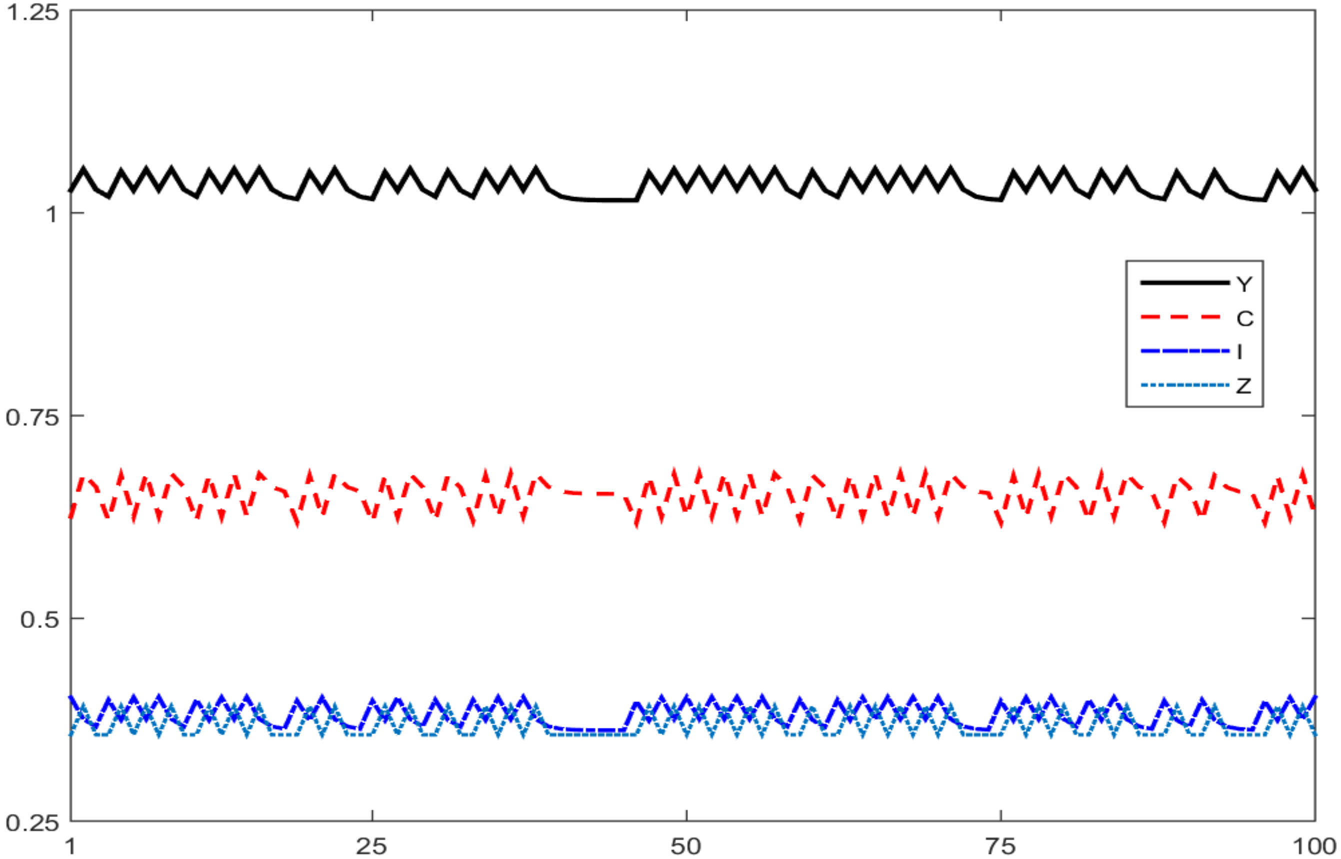


Simulated series with const. probability: π=0.5

Simulated output (Y), consumption (C) and investment (I) normalized by steady state output



Lower volatility if probability of investment bust rises once investment/output ration Z_t crosses threshold.



Simulated series with state-contingent probability of bust:

$\pi_t = 0.5$ for $\alpha\beta + \Delta = 0.356 \leq Z_t \leq 0.36$ & $\pi_t = 1 - 10^{-100}$ for $Z_t > 0.36$

Table 1. Long-Plosser model with bubbles: predicted business cycle statistics

	Standard dev. %			Corr. with Y		Autocorr.			Mean (% deviation from SS)			
	Y	C	I	C	I	Y	C	I	Y	C	I	Z
	(1)	*(2)*	*(3)*	*(4)*	*(5)*	*(6)*	*(7)*	*(8)*	*(9)*	*(10)*	*(11)*	*(12)*
(a) Specification I: $Z_t^L = \alpha\beta + \Delta$												
π_t=0.5	11.72	100.19	33.48	-0.42	0.62	0.62	0.47	0.62	13.49	-7.62	53.31	31.15
$\pi_t \cong 1$ for $z_t > 0.36$	1.33	3.51	3.82	0.77	-0.26	-0.26	-0.66	-0.26	3.27	-0.13	9.71	6.25
(b) US Data (from King and Rebelo (1999))												
	1.81	1.35	5.30	0.88	0.80	0.88	0.80	0.87				

Note: all business statistics pertain to HP-filtered logged variables.

Example II: RBC model with incomplete capital depreciation & endogenous labor

$U(C_t, L_t) = \ln(C_t) + \Psi \cdot \ln(1 - L_t), \quad \Psi > 0$, L_t: hours worked

$C_t + K_{t+1} = Y_t + (1-\delta)K_t$, $\quad Y_t = \theta (K_t)^\alpha (L_t)^{1-\alpha}$

- FOCs: $C_t \Psi / (1 - L_t) = (1-\alpha)\theta (K_t)^\alpha (L_t)^{-\alpha}$

$$E_t \beta \{C_t / C_{t+1}\}(\alpha\theta (K_{t+1})^{\alpha-1}(L_{t+1})^{1-\alpha} + 1 - \delta) = 1$$

- Using static efficiency conditions can express *C* & *L* and functions of capital & TFP:

$C_t = \gamma(K_{t+1}, K_t)$, $L_t = \eta(K_{t+1}, K_t)$

Can writer Euler equation as:

$$E_t [\beta \{\gamma(K_{t+1}, K_t) / \gamma(K_{t+2}, K_{t+1})\}(\alpha\theta (K_{t+1})^{\alpha-1}(\eta(K_{t+2}, K_{t+1}))^{1-\alpha} + 1 - \delta)] = 1$$

Euler equation:

$$E_t H(K_{t+2}, K_{t+1}, K_t) = 1$$

No bubble solution (TVC): described by policy function $K_{t+1} = \lambda(K_t)$

so that $E_t H(\lambda(\lambda(K_t)), \lambda(K_t), K_t) = 1$

Consider **bubble equilibria** such that, for any *t*, K_{t+1} takes one of two values $K_{t+1} \in \{K_{t+1}^{L}, K_{t+1}^{H}\}$ with exogenous probabilities π and $1-\pi$, where $K_{t+1}^{L}=\lambda(K_t)e^{\Delta}$;
$\Delta>0$: small positive constant

'L' is 'bust' state, in which capital stock set at t reverts to value close to 'no-bubble' decision rule

Euler equation
$E_t H(K_{t+2}, K_{t+1}, K_t) = 1$
becomes:
$$\pi H(\lambda(K_{t+1})e^{\Delta}, K_{t+1}, K_t) + (1-\pi)\cdot H(K_{t+2}^{H}, K_{t+1}, K_t) = 1$$

Economy evolves as follows:

At date t: random draw (with probab. π, 1-π) determines $K_{t+1} \in \{K_{t+1}^L, K_{t+1}^H\}$ where $K_{t+1}^L = \lambda(K_t)e^{\Delta}$

Euler equation between t and t+1 determines K_{t+2}^H:

$$\pi H(\lambda(K_{t+1})e^{\Delta}, K_{t+1}, K_t) + (1-\pi) \cdot H(K_{t+2}^H, K_{t+1}, K_t) = 1$$

Etc. in all subsequent periods.

See paper for: ● Existence proof of sunspot equilibrium: need Δ>0. Then $K_{t+1}^L < K_{t+1}^H$

● Analysis with stochastic TFP

Numerical simulations

β=0.99; α=1/3; δ=0.025;

Labor supply elasticity (at steady state) = 1.

- Log utility (unit risk aversion, RA): $\ln(C_t)$
- 'High Risk Aversion' utility: $\ln(C_t - \overline{C}),\ \overline{C} > 0$

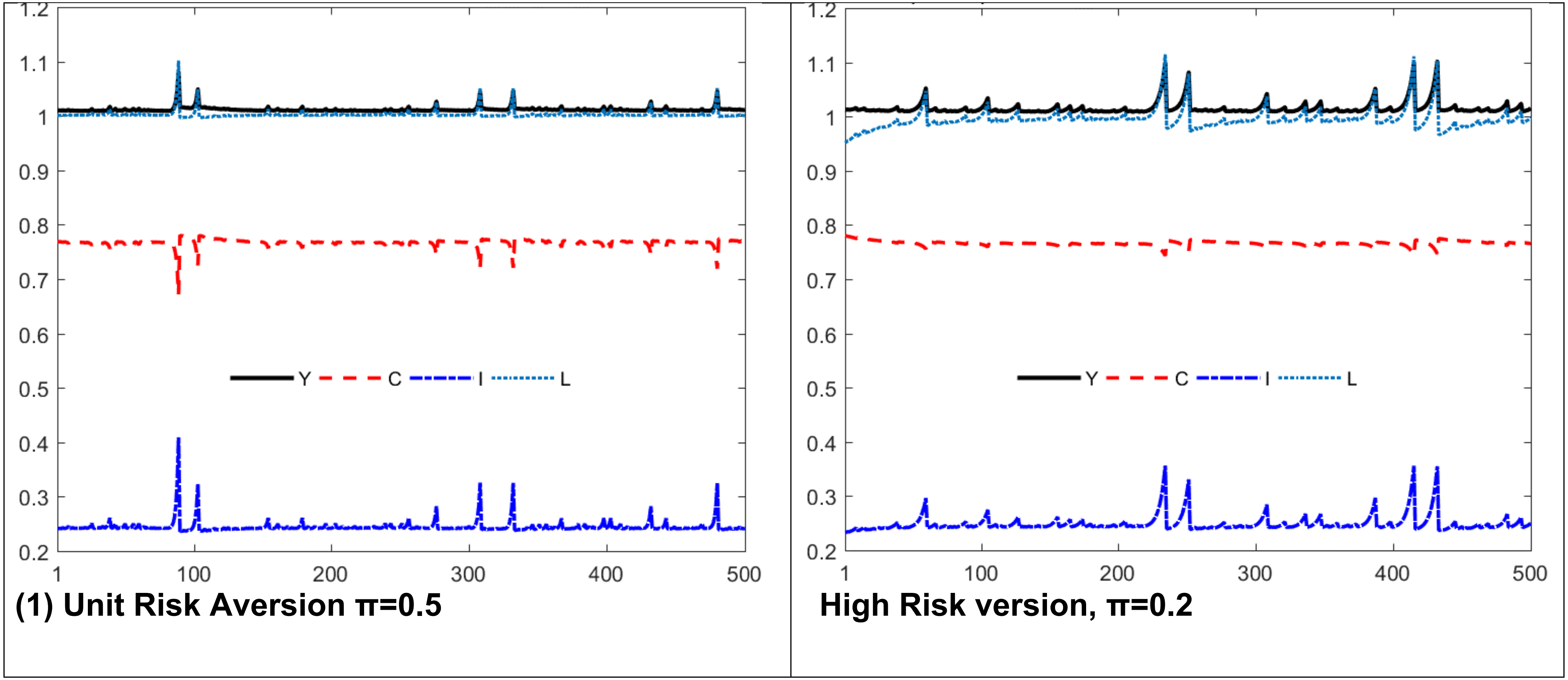
Parameters of bubble process:

Δ=0.001

Bust probability: π=0.5, π=0.2.

RBC model (incomplete capital deprec.) with bubbles: predicted business cycle statistics

	Unit Risk aversion		High RA		
	π=0.5	π=0.2	π=0.5	π=0.2	**Data**
	(1)	*(2)*	*(3)*	*(4)*	*(5)*
Standard deviations [in %]					
Y	0.49	1.16	0.68	1.43	1.81
C	1.08	2.63	0.29	0.61	1.35
I	4.29	9.38	3.22	6.51	5.30
L	0.74	1.73	1.04	2.18	1.79
Correlations with GDP					
C	-0.97	-0.95	-0.99	-0.98	0.88
I	0.98	0.96	0.99	0.99	0.80
L	0.99	0.97	0.99	0.99	0.88
Autocorrelations					
Y	0.36	0.63	0.35	0.62	0.84
C	0.33	0.60	0.35	0.62	0.80
I	0.36	0.63	0.37	0.64	0.87
L	0.34	0.61	0.35	0.62	0.88
Means [% deviation from steady state]					
Y	1.41	2.80	1.25	2.12	--
C	0.73	1.39	0.33	0.55	--
I	3.62	7.33	4.22	7.19	--
L	0.36	0.74	-0.02	-0.02	--
Mean (capital income – investment)/GDP [in %]					
	9.12	8.75	8.93	8.54	13.42
Fraction of periods with (capital income > investment) [in %]					
	99.2	96.3	99.5	97.7	100



Non-linear RBC model (incompl. capital depreciation) driven by bubbles

Simulated GDP, C and I series normalized by steady state GDP. Hours worked (L) normalized by steady state hours.

CONCLUSIONS

- Stationary sunspot equilibria exist in standard *non-linear* DSGE models, even when the linearized versions of those models have unique solutions.
- In the sunspot equilibria considered here, the economy temporarily diverges from the no-sunspots trajectory, before abruptly reverting towards that trajectory.
- In contrast to rational bubbles in linear models (Blanchard (1979)), the bubbles considered here are stationary--their expected path does not explode to infinity.

ADDITIONAL MATERIAL

Blanchard (1979):

$$E_t y_{t+1} = \lambda \cdot y_t, \quad \lambda > 1 \quad \Rightarrow \quad y_{t+1} = \lambda \cdot y_t + \varepsilon_{t+1}, \quad E_t \varepsilon_{t+1} = 0$$

How non-linearity may generate stationary bubble:

Assume: $E_t \exp(z_{t+1} - \lambda z_t) = a, \quad \lambda > 1, \; a > 0$

$\Rightarrow \exp(z_{t+1} - \lambda z_t) = a + \eta_{t+1}$ with $E_t \eta_{t+1} = 0$

$\Rightarrow z_{t+1} = \lambda z_t + \log(a + \eta_{t+1})$. Let $y_t \equiv z_t + \ln(a)/(\lambda - 1), \; \varepsilon_{t+1} \equiv \eta_{t+1}/a$

$$\Rightarrow y_{t+1} = \lambda \cdot y_t + \ln(1 + \varepsilon_{t+1}), \quad E_t \varepsilon_{t+1} = 0$$

y_{t+1} is **<u>concave</u>** in $\varepsilon_{t+1} \Rightarrow E_t y_{t+1} < \lambda \cdot y_t$

Let $\underline{\varepsilon_{t+1}} \in \{-\bar{\varepsilon}_t; \; \bar{\varepsilon}_t \pi/(1-\pi)\}$ with prob. $\pi, 1-\pi$. $\bar{\varepsilon}_t > 0$

Set $\bar{\varepsilon}_t \in [0,1)$ so that $y_{t+1} = \lambda \cdot y_t + \ln(1 - \bar{\varepsilon}_t) = \Delta < 0$

$y_{t+1} = y_{t+1}^H \equiv \lambda \cdot y_t + \ln\{1 + [1 - \exp(\Delta - \lambda \cdot y_t)] \pi/(1-\pi)\}$ with prob. $1-\pi$

$y_{t+1} = \Delta$ with probability π