Identifying Indicators of Systemic Risk

Benny Hartwig Christoph Meinerding Yves Schüler

Deutsche Bundesbank

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Objective of macroprudential policy: Address systemic risk

Which indicators should be used to inform policy?

- Disagreement/uncertainty about which indicators can be used to measure systemic risk

This paper tries to fill this gap as objectively as possible:

- Based on statistical hypothesis tests
- Discriminate between variables we should or should not use for policy.

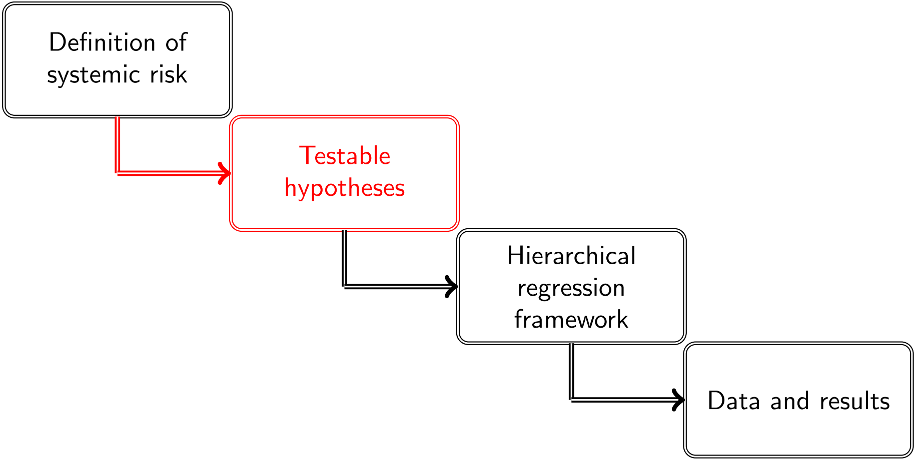
The definition

In their report to the G20 finance ministers in 2009, IMF, BIS, and FSB define systemic risk as a

> **"risk of disruption to financial services that is (i) caused by an impairment of all or parts of the financial system and (ii) has the potential to have serious negative consequences for the real economy"**

Our goals and contributions:

- Derive testable hypotheses that can classify a variable as an indicator of systemic risk
- Remain objective, stick to definition as closely as possible
- Present parsimonious testing framework for these hypotheses
- Apply test to set of candidate indicators (currently U.S. data)


Definition of systemic risk
Testable hypotheses
Hierarchical regression framework
Data and results

Hypothesis 1

"**risk of disruption to financial services** *that is (i)* **caused by an impairment** *of all or parts* **of the financial system** *and (ii) has the potential to have serious negative consequences for the real economy*"

- "Risk": Today's probability of an event in the future.
  - How far into the future? ⇒ time dimension of systemic risk
- Which event? "Disruption to financial services caused by ..."

Hypothesis 1:

⇒ **Indicator needs to measure probability of a future event that qualifies as "disruption to financial services caused by an impairment of the financial system"**

Hypothesis 2

"risk of disruption to financial services that is (i) caused by an impairment of all or parts of the financial system and (ii) has the **potential to have serious negative consequences for the real economy***"*

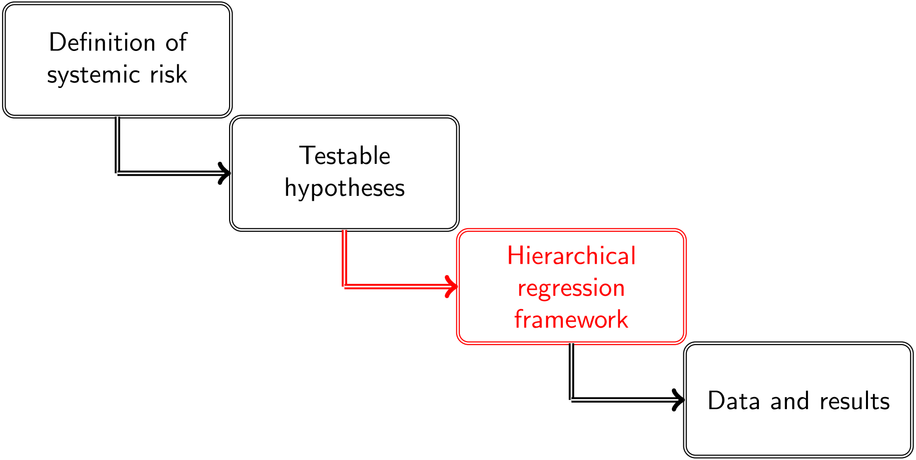
- Not all potential disruptions need to feed into systemic risk

⇒ Disruption must affect the real economy.

- "Potential": affects distribution of real economic variables.
- "Serious negative consequences": left tail of the distribution.

Hypothesis 2:

⇒ **Risk of disruption must be positively correlated with tail risk for the real economy.**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Stage 1

Indicator needs to measure probability of a future event that qualifies as "disruption to financial services caused by an impairment of the financial system".

Test of Hypothesis 1:

Draw on early-warning literature on financial crises

- Indicator (x_t) should predict disruption defined by crisis dummies (d_t) – assuming disruptions can be detected ex post
- Logit regression:

$$\text{logit}(\pi_{t,t+h}) = \alpha + \sum_{k=0}^{K} \beta_k x_{t-k} \quad (1)$$

  - $\text{logit}(\pi_{t,t+h}) = \ln(\pi_{t,t+h}/(1 - \pi_{t,t+h}))$
  - $\pi_{t,t+h} = P(d_{t+h} = 1 | \text{info}_t)$
  - K: chosen according to Bayes Information Criterion (BIC)
  - for various horizons h

- Candidate passes test if $\exists k$ s.t. $\beta_k \neq 0$ (likelihood ratio test)

Stage 2

Risk of disruption must be positively correlated with tail risk for the real economy.

Test of Hypothesis 2:

Draw on growth-at-risk literature

- Risk of disruption ($\widehat{\pi}_{t,t+h}$, Stage 1) – not necessarily x_t itself – should explain movement of macro downside risk.
- Quantile regression at quantile $\tau = 5\%$:

$$y_{t+h} = \gamma_\tau + \delta_\tau \widehat{\pi}_{t,t+h} + \boldsymbol{\omega}_\tau \mathbf{z}_t + \varepsilon_{t+h} \tag{2}$$

  - y_{t+h}: GDP growth at $t+h$.
  - $\mathbf{z}_t$: controls (here: lagged GDP growth)
  - difference to linear regression:
    ε not normal, objective function not sum-of-squared-errors

- Candidate passes test if $\delta_\tau < 0$
  (one-sided t-test with adjusted standard errors).

Explicit vs. implicit indicators

Quantile vs. linear regression

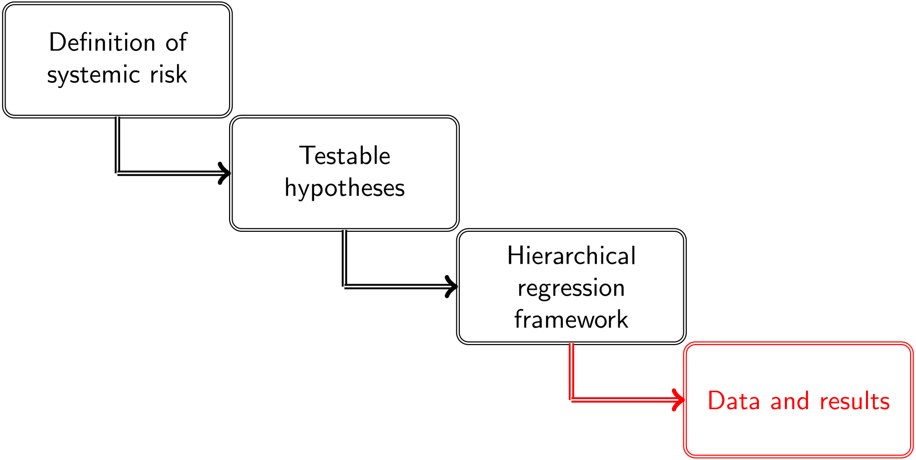
- Both regressions can explain time variation in tails
  - Quantile regression: *explicit* modeling of the tail
  - Linear regression: *implicitly* via time variation in the center
- Definition requires "serious negative consequences".
  - **Explicit indicator of systemic risk:**
    → Passes Stage 1 and Stage 2 for quantile regression
  - **Implicit indicator of systemic risk:**
    → Passes Stage 1 and Stage 2 for linear regression
- Ratio of quantile and linear coefficients informative about "seriousness" (i.e. degree of nonlinearity in tails).

Adjusting standard errors

Hierarchical test framework poses challenge for inference

Predicted probability from Stage 1 is a generated regressor
$\Rightarrow$ **Adjust standard errors of Stage 2**

1. Starting point: maximum likelihood framework of Murphy and Topel (JBES 1985 & 2002)
2. Potentially error terms on Stage 2 not identically distributed $\rightarrow$ extend general formulas to quasi-MLE ▸ Technical details
3. Application of formulas to case "logit + quantile regression" based on QMLE framework in Komunjer (2005) ▸ Technical details
4. Application of formulas to case "logit + linear regression" straightforward ▸ Technical details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Candidate indicators of systemic risk

1. **Basel III credit-to-GDP gap**
   - leading indicator for triggering CCyB in 73 countries
   - based on total credit to private non-financial sector
2. **Composite financial cycle – Schüler et al. (2017)**
   - common fluctuations in credit and asset prices
   - time-varying linear combination of standardized growth rates
   - based on growth rates (not levels)
3. **National Financial Conditions Index (NFCI)**
   - principal component of 105 financial variables (related to credit risk, amount of credit, volatility, leverage)
   - used in the growth-at-risk paper of Adrian et al. (AER, 2019)
4. **Gilchrist Zakrajsek (2012) corporate bond credit spread**
   - extracted from micro data, only for US
   - perhaps more a recession indicator?
5. **Term spread (10y minus 3m)**
   - typical recession indicator

Candidates transformed to quarterly/semi-annual by averaging (if necessary)

Dummy variables for disruption to financial services

1. **Romer Romer (AER, 2017)**
   - disruption to credit supply (on a 0 to 15 scale)
   - based on narrative approach
   - very granular, also detects smaller disruptions
   - available for 24 OECD countries
   - we map 0-15 scale into a 0-1 dummy
   - semi-annual, 1973H1–2012H2
2. **Laeven Valencia (IMF, 2018)**
   - quantitative approach, based on a set of indicator variables exceeding certain thresholds
   - 0-1 dummy variable
   - available for 165 countries
   - quarterly, 1973Q1–2015Q4

Results (exemplary case)

- Horizon: 1 year ahead
- Crisis dummies: Romer Romer (2017)

	Stage 1	Stage 2	
	LR-test statistic	linear coeff.	5% quantile coefficient
Credit-to-GDP	**29.2** (3)	**-2.8**	1.6
Financial cycle	**15.8** (3)	**-6.1**	**-11.8**
NFCI	0.0 (0)	-106.9	-684.0
GZ-Spread	**7.2** (0)	-1.7	2.1
Term spread	0.5 (0)	-25.3	-40.1

Bold figures: significance at the 10% level.
Bold and underlined figures: significance at the 5% level.
Dark gray area: explicit indicator of systemic risk that passes all stages at least at the 5% level.
Light gray area: explicit indicator of systemic risk that passes all stages at least at the 10% significance level.
In parentheses: number of lags in Stage 1 (determined via BIC).
In Stage 2: Two lags of semi-annual GDP growth as controls.

- Credit-to-GDP is an implicit indicator
- FCycle is an explicit indicator
- Severity (ratio of coefficients) for FCycle ≈ 2
- GZ spread fails on Stage 2
- NFCI and term spread fail on Stage 1

Results (Romer Romer 2017 dummies)

	Stage 1 LR-test	Stage 2 linear	5%	Stage 1 LR-test	Stage 2 linear	5%	Stage 1 LR-test	Stage 2 linear	5%
	1 quarter ahead			1/2 year ahead			1 year ahead		
Credit-to-GDP				✓	**-1.9**	**-3.4**	✓	**-2.8**	1.6
Financial cycle				✓	**-4.2**	**-10.9**	✓	**-6.1**	**-11.8**
NFCI				✗			✗		
GZ-Spread				✓	-2.6	7.3	✓	-1.7	2.1
Term spread				✗			✗		
	1.5 year ahead			2 years ahead			3 years ahead		
Credit-to-GDP	✓	**-2.8**	-2.2	✓	**-2.9**	-3.5	✓	**-3.3**	-1.0
Financial cycle	✓	**-5.5**	**-10.6**	✓	**-6.1**	**-12.0**	✗		
NFCI	✗			✗			✗		
GZ-Spread	✓	-1.6	17.2	✓	-1.3	13.0	✓	**-4.3**	13.4
Term spread	✗			✗			✗		

- Credit-to-GDP: passes test up to 3 years ahead (implicit)
- Financial cycle: passes test up to 2 years ahead (explicit)
- NFCI: largely fails (passes for 1 quarter ahead only)
- GZ-spread: passes test for 3 years ahead (implicit)
- Term spread: fails

Results (Laeven Valencia 2018 dummies)

	Stage 1 LR-test	Stage 2 linear	Stage 2 5%	Stage 1 LR-test	Stage 2 linear	Stage 2 5%	Stage 1 LR-test	Stage 2 linear	Stage 2 5%
	1 quarter ahead			1/2 year ahead			1 year ahead		
Credit-to-GDP	✓	**-3.0**	**-5.6**	✓	**-4.0**	**-5.9**	✓	**-4.3**	**-6.1**
Financial cycle	✓	**-15.3**	**-32.7**	✓	**-11.3**	**-21.3**	✓	**-5.2**	**-24.0**
NFCI	✓	**-19.2**	**-34.7**	✗			✗		
GZ-Spread	✓	**-5.5**	-7.4	✓	**-5.4**	2.6	✗		
Term spread	✗			✗			✗		
	1.5 year ahead			2 years ahead			3 years ahead		
Credit-to-GDP	✓	**-4.0**	**-4.7**	✓	**-3.0**	**-10.4**	✓	**-1.4**	3.8
Financial cycle	✓	**-6.7**	-8.7	✓	**-4.7**	6.9	✓	-1.7	2.0
NFCI	✗			✓	2.7	17.2	✓	-4.0	-1.8
GZ-Spread	✗			✗			✗		
Term spread	✓	**-12.6**	**-58.0**	✓	**-10.5**	**-31.1**	✗		

- Credit-to-GDP: now also explicit
- Financial cycle: explicit only up to 1 year ahead
- NFCI: passes for 1 quarter ahead only
- GZ-spread: passes test up to 0.5 years ahead
- Term spread: passes test 1.5-2 years ahead
- **Results point towards nonlinearity** (linear regression underestimates effects on tails)

Conclusion

Contributions

1. Operationalize definition of systemic risk of IMF, BIS, FSB
2. Derive testable hypotheses + two-stage hierarchical test to identify indicators of systemic risk
3. Combine early-warning literature and growth-at-risk

Results

- Measures capturing procyclicality of financial system qualify as indicators of systemic risk (up to 3 years ahead)
- Variables capturing spillovers and interlinkages don't
- Results point towards nonlinearity
- Results support theoretical channels like leverage cycles

Extension towards other countries, candidate variables, or crisis dummies is straightforward

Thank you very much!

Technical details: Standard errors with generated regressors (1/3)

Back

Theorem (Asymptotic distribution of two-step QMLE)

Suppose our model consists of the two marginal distributions $f_1(y_1|x_1, \theta_1)$ *and* $f_2(y_2|x_1, x_2, \theta_1, \theta_2)$. *The estimation proceeds in two steps:*

1. *Estimate* θ_1 *by maximum likelihood in model 1:* $L_1(\theta_1) = \prod_{t=1}^{T} f_1(y_{1t}|x_{1t}, \theta_1)$.
2. *Estimate* θ_2 *by maximum likelihood in model 2, with* $\hat{\theta}_1$ *for* θ_1*, i.e. as if* θ_1 *was known:* $L_2(\theta_1, \theta_2) = \prod_{t=1}^{T} f_2(y_{2t}|x_{1t}, x_{2t}, \theta_1, \theta_2)$.

If the standard regularity conditions for both log-likelihood functions hold and if the quasi maximum likelihood estimate of θ_2 *is consistent, then the MLE of* θ_2 *is asymptotically normally distributed with asymptotic covariance matrix ...*

Technical details: Standard errors with generated regressors (2/3)

Theorem (Asymptotic distribution of two-step QMLE)

$$V_2 = \frac{1}{T}(-H_{22}^{(2)})^{-1}\Sigma_{22}(-H_{22}^{(2)})^{-1}$$
$$+ \frac{1}{T}(-H_{22}^{(2)})^{-1}\left(H_{21}^{(2)}(-H_{11}^{(1)})^{-1}H_{21}^{(2)\prime} + \Sigma_{21}(-H_{11}^{(1)})^{-1}H_{21}^{(2)\prime} + H_{21}^{(2)}(-H_{11}^{(1)})^{-1}\Sigma_{12}\right)(-H_{22}^{(2)})^{-1}$$

where

$$\Sigma_{22} = E\left[\frac{1}{T}\frac{\partial \ln L_2(\theta_1,\theta_2)}{\partial\theta_2}\frac{\partial \ln L_2(\theta_1,\theta_2)}{\partial\theta_2'}\right], \quad \Sigma_{21} = E\left[\frac{1}{T}\frac{\partial \ln L_2(\theta_1,\theta_2)}{\partial\theta_2}\frac{\partial \ln L_1(\theta_1)}{\partial\theta_1'}\right],$$
$$\Sigma_{12} = E\left[\frac{1}{T}\frac{\partial \ln L_1(\theta_1)}{\partial\theta_1}\frac{\partial \ln L_2(\theta_1,\theta_2)}{\partial\theta_2'}\right], \quad H_{11}^{(1)} = E\left[\frac{1}{T}\frac{\partial^2 \ln L_1(\theta_1)}{\partial\theta_1\partial\theta_1'}\right],$$
$$H_{22}^{(2)} = E\left[\frac{1}{T}\frac{\partial^2 \ln L_2(\theta_1,\theta_2)}{\partial\theta_2\partial\theta_2'}\right], \quad H_{21}^{(2)} = E\left[\frac{1}{T}\frac{\partial^2 \ln L_2(\theta_1,\theta_2)}{\partial\theta_2\partial\theta_1'}\right].$$

Technical details: Standard errors with generated regressors (3/3)

Theorem (Asymptotic distribution of two-step QMLE)

The estimate $\hat{V}_2$ is given by

$$\hat{V}_2 = (-\hat{H}_{22}^{(2)})^{-1}[\hat{\Sigma}_{22} + \hat{H}_{21}^{(2)}(-\hat{H}_{11}^{(1)})^{-1}\hat{H}_{21}^{(2)'} + \hat{\Sigma}_{21}(-\hat{H}_{11}^{(1)})^{-1}\hat{H}_{21}^{(2)'} + \hat{H}_{21}^{(2)}(-\hat{H}_{11}^{(1)})^{-1}\hat{\Sigma}_{12}](-\hat{H}_{22}^{(2)})^{-1}$$

where $\hat{\Sigma}_{22}, \hat{\Sigma}_{21}$ and $\hat{\Sigma}_{12}$ are the typical BHHH estimators

$$\hat{\Sigma}_{22} = \sum_{t=1}^{T} \frac{\partial \ln f_{2t}}{\partial \hat{\theta}_2} \frac{\partial \ln f_{2t}}{\partial \hat{\theta}_2'}, \quad \hat{\Sigma}_{21} = \sum_{t=1}^{T} \frac{\partial \ln f_{2t}}{\partial \hat{\theta}_2} \frac{\partial \ln f_{1t}}{\partial \hat{\theta}_1'}, \quad \hat{\Sigma}_{12} = \sum_{t=1}^{T} \frac{\partial \ln f_{1t}}{\partial \hat{\theta}_1} \frac{\partial \ln f_{2t}}{\partial \hat{\theta}_2'}$$

and the $\hat{H}_{11}$, $\hat{H}_{22}$ and $\hat{H}_{21}$ may be computed as expected Hessians

$$\hat{H}_{11}^{(1)} = \sum_{t=1}^{T} E\left[\frac{\partial \ln^2 f_{1t}}{\partial \hat{\theta}_1 \partial \hat{\theta}_1'}\right], \quad \hat{H}_{22}^{(2)} = \sum_{t=1}^{T} E\left[\frac{\partial \ln^2 f_{2t}}{\partial \hat{\theta}_2 \partial \hat{\theta}_2'}\right], \quad \hat{H}_{21}^{(2)} = \sum_{t=1}^{T} E\left[\frac{\partial \ln^2 f_{2t}}{\partial \hat{\theta}_2 \partial \hat{\theta}_1'}\right].$$

Technical details: Application to Logit + Linear Regression

Back

Stage 1: Logit model

$$P(y_{1t} = 1) = \Lambda(x_{1t}\theta_1)$$

where $\Lambda(x_t\theta) = \frac{\exp(x_t\theta)}{1+\exp(x_t\theta)}$. The log-likelihood is

$$\ln L_1(\theta_1) = \sum_{t=1}^{T} \ln f_1(y_{1t}|x_{1t}, \theta_1) = \sum_{t=1}^{T} \left[(1 - y_{1t}) \ln[(1 - \Lambda(x_{1t}\theta_1))] + y_{1t} \ln[\Lambda(x_{1t}\theta_1)]\right].$$

Stage 2: Linear regression model

$$E(y_{2t}|x_{1t}, x_{2t}, \theta_1, \theta_2) = x_{2t}\beta + \sum_{k=0}^{p} \Lambda(x_{1t-k}\theta_1)\gamma_k = z_t\theta_2.$$

The log-likelihood is

$$\ln L_2(\theta_1, \theta_2) = \sum_{t=1}^{T} \ln f_2(y_{2t}|x_{1t}, x_{2t}, \theta_1, \theta_2) = -\frac{T}{2}\ln(2\pi) - \frac{T}{2}\ln(\sigma^2) - \sum_{t=1}^{T} \frac{1}{2\sigma^2} u_{2t}^2$$

where $u_{2t} = y_{2t} - z_t\theta_2$.
Derivatives of the log-likelihood w.r.t. θ_1 and θ_2 are straightforward.

Technical details: Application to Logit + Linear Regression

Back

Inputs for the corrected asymptotic covariance matrix:

$$\Sigma_{22} = E\left(\frac{1}{T}\left(\frac{1}{\sigma^2}\right)^2\sum_{t=1}^{T} u_{2t}^2 z_t' z_t\right), \qquad \Sigma_{21} = E\left(\frac{1}{T}\frac{1}{\sigma^2}\sum_{t=1}^{T} u_{1t}u_{2t}z_t' x_{1t}\right)$$

$$\Sigma_{12} = E\left(\frac{1}{T}\frac{1}{\sigma^2}\sum_{t=1}^{T} u_{1t}u_{2t}x_{1t}' z_t\right), \qquad H_{11}^{(1)} = E\left(-\frac{1}{T}\sum_{t=1}^{T} x_{1t}' x_{1t}\Lambda(x_{1t}\theta_1)(1-\Lambda(x_{1t}\theta_1))\right)$$

$$H_{21}^{(2)} = E\left(-\frac{1}{T}\frac{1}{\sigma^2}\sum_{t=1}^{T} z_t' n_t\right), \qquad H_{22}^{(2)} = E\left(-\frac{1}{T}\frac{1}{\sigma^2}\sum_{t=1}^{T} z_t' z_t\right)$$

with

$$n_t = \frac{\partial \sum_{j=1}^{k_2} z_{tj}\theta_{2j}}{\partial \theta_1'} = \sum_{k=0}^{p} x_{1t-k}\Lambda(x_{1t-k}\theta_1)(1-\Lambda(x_{1t-k}\theta_1))\gamma_k$$

Technical details: Application to Logit + Linear Regression

Back

Empirical gradients for the BHHH-Type estimators:

$$\frac{\partial \ln f_1}{\partial \hat{\theta}_1} = x_{1t}' \hat{u}_{1t}, \quad \frac{\partial \ln f_2}{\partial \hat{\theta}_2} = \frac{1}{\hat{\sigma}^2} \hat{z}_t' \hat{u}_{2t}$$

Expected Hessians

$$E\left[\frac{\partial^2 \ln f_1}{\partial \hat{\theta}_1 \partial \hat{\theta}_1'}\right] = -x_{1t}' x_{1t} \Lambda(x_{1t}\hat{\theta}_1)(1 - \Lambda(x_{1t}\hat{\theta}_1),$$

$$E\left[\frac{\partial^2 \ln f_2}{\partial \hat{\theta}_2 \partial \hat{\theta}_1'}\right] = -\frac{1}{\hat{\sigma}^2} \hat{z}_t' \hat{n}_t, \qquad E\left[\frac{\partial^2 \ln f_2}{\partial \hat{\theta}_2 \partial \hat{\theta}_2'}\right] - \frac{1}{\hat{\sigma}^2} \hat{z}_t' \hat{z}_t.$$

Technical details: Application to Logit + Quantile Regression

Back

Stage 1: Logit model

$$P(y_{1t} = 1) = \Lambda(x_{1t}\theta_1)$$

where $\Lambda(x_t\theta) = \frac{\exp(x_t\theta)}{1+\exp(x_t\theta)}$. The log-likelihood is

$$\ln L_1(\theta_1) = \sum_{t=1}^{T} \ln f_1(y_{1t}|x_{1t},\theta_1) = \sum_{t=1}^{T} [(1-y_{1t})\ln[(1-\Lambda(x_{1t}\theta_1))] + y_{1t}\ln[\Lambda(x_{1t}\theta_1)]] \,.$$

Stage 2: Quantile regression model

$$Q_\tau(y_{2t}|x_{1t},x_{2t},\theta_1,\theta_2^\tau) = x_{2t}\beta^\tau + \sum_{k=0}^{p} \Lambda(x_{1t-k}\theta_1)\gamma_k^\tau = z_t\theta_2^\tau \,.$$

Log-likelihood function (Komunjer 2005):

$$\ln L_2(\theta_1,\theta_2^\tau) = \sum_{t=1}^{T} \quad -(1-\tau)\left(\frac{1}{\tau(1-\tau)}(z_t\theta_2^\tau - y_{2t})1_{\{y_{2t}\le z_t\theta_2^\tau\}}\right)$$
$$+\tau\left(\frac{1}{\tau(1-\tau)}(z_t\theta_2^\tau - y_{2t})1_{\{y_{2t}> z_t\theta_2^\tau\}}\right)$$

Derivatives of log-likelihood: only exist in the “distributional” (generalized) sense.

Technical details: Application to Logit + Quantile Regression

Back

Inputs for the corrected asymptotic covariance matrix:

$$
\begin{aligned}
\Sigma_{22} &= E\left(\frac{1}{T}\sum_{t=1}^{T} g_{2t}^{(2)} g_{2t}^{(2)\prime}\right) = \frac{1}{\tau(1-\tau)} E\left[\frac{1}{T}\sum_{t=1}^{T} z_t' z_t\right] \\
\Sigma_{21} &= E\left(\frac{1}{T}\sum_{t=1}^{T} g_{2t}^{(2)} g_{1t}^{(1)\prime}\right) = \frac{1}{\tau(1-\tau)} E\left(\frac{1}{T}\sum_{t=1}^{T} u_{1t}(\tau - 1_{\{y_{2t}\leq z_t\theta_2^\tau\}}) z_t' x_{1t}\right) \\
\Sigma_{12} &= E\left(\frac{1}{T}\sum_{t=1}^{T} g_{1t}^{(1)} g_{2t}^{(2)\prime}\right) = \frac{1}{\tau(1-\tau)} E\left(\frac{1}{T}\sum_{t=1}^{T} u_{1t}(\tau - 1_{\{y_{2t}\leq z_t\theta_2^\tau\}}) x_{1t}' z_t\right) \\
H_{11}^{(1)} &= E\left(\frac{1}{T}\sum_{t=1}^{T} g_{11t}^{(1)}\right) = -E\left(\frac{1}{T}\sum_{t=1}^{T} x_{1t}' x_{1t} \Lambda(x_{1t}\theta_1)(1-\Lambda(x_{1t}\theta_1))\right) \\
H_{21}^{(2)} &= E\left(\frac{1}{T}\sum_{t=1}^{T} g_{21t}^{(2)}\right) = -\frac{1}{\tau(1-\tau)} E\left(\frac{1}{T}\sum_{t=1}^{T} z_t' n_t f_{y_{2t}|z_t\theta_2^\tau}(z_t\theta_2^\tau)\right) \\
H_{22}^{(2)} &= E\left(\frac{1}{T}\sum_{t=1}^{T} g_{22t}^{(2)}\right) = -\frac{1}{\tau(1-\tau)} E\left(\frac{1}{T}\sum_{t=1}^{T} z_t' z_t f_{y_{2t}|z_t\theta_2^\tau}(z_t\theta_2^\tau)\right)
\end{aligned}
$$

Technical details: Application to Logit + Quantile Regression

Back

Empirical gradients for the BHHH-Type estimators

$$\frac{\partial \ln f_1}{\partial \hat{\theta}_1} = x'_{1t}\hat{u}_{1t}, \quad \frac{\partial \ln f_2}{\partial \hat{\theta}_2} = \hat{z}'_t(\tau - 1_{\{y_{2t} \leq \hat{z}_t \hat{\theta}_2^{\tau}\}})$$

Expected Hessians

$$E\left[\frac{\partial^2 \ln f_1}{\partial \hat{\theta}_1 \partial \hat{\theta}'_1}\right] = -x'_{1t}x_{1t}\Lambda(x_{1t}\hat{\theta}_1)(1 - \Lambda(x_{1t}\hat{\theta}_1),$$

$$E\left[\frac{\partial^2 \ln f_2}{\partial \hat{\theta}_2 \partial \hat{\theta}'_1}\right] = -\frac{1}{\tau(1-\tau)}\hat{z}'_t\hat{n}_t\hat{f}_{y_{2t}|\hat{z}_t\hat{\theta}_2^{\tau}}(\hat{z}_t\hat{\theta}_2^{\tau}), \quad E\left[\frac{\partial^2 \ln f_2}{\partial \hat{\theta}_2 \partial \hat{\theta}'_2}\right] = -\frac{1}{\tau(1-\tau)}\hat{z}'_t\hat{z}_t\hat{f}_{y_{2t}|\hat{z}_t\hat{\theta}_2^{\tau}}(\hat{z}_t\hat{\theta}_2^{\tau}).$$

We estimate the density of the errors using the kernel method of Powell (1991):

$$\hat{f}_{y_{2t}|\hat{z}_t\hat{\theta}_2^{\tau}}(\hat{z}_t\hat{\theta}_2^{\tau}) = \frac{1}{2c_T}1(|\hat{u}_{2t}| < c_T)$$

where

$$c_T = \kappa(\Phi^{-1}(\tau + h_T) - \Phi^{-1}(\tau - h_T))$$

κ is a robust scale estimate and h_T is chosen according to Hall and Sheather (1988).