Determinants of the term structure of interest rates – approaches to combining arbitrage-free models and monetary macroeconomics

The term structure of interest rates represents the relationship between the maturities and the yields of bonds. While short-term interest rates are influenced crucially by monetary policy, longer-term interest rates mainly reflect market players' expectations of future macroeconomic developments. Interest rates of different maturities do not move independently of each other, however. Rather, they are linked by the condition of absence of arbitrage, which means that the term structure must not allow any trading strategy which permits risk-free investment profits from investment in bonds of differing maturities. Modern term structure models link this key concept from the finance literature to explanatory approaches from macroeconomics. This article presents the basic idea of such combined modelling using the German term structure as an illustration. It identifies how the term structure reacts to inflationary and business cycle movements and calculates the level of the risk premiums contained in bond yields.

Basic concepts and shape of the term structure over time

The nominal term structure reflects the relationship between the maturity of a bond and

Term structure based on Federal Government issues

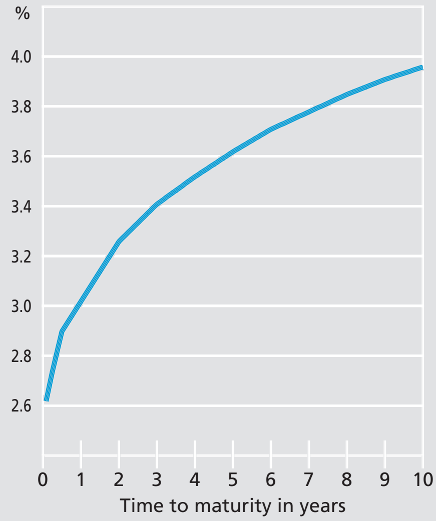
the corresponding rate of interest.[1] The securities issued by the Federal Government have maturities at issue ranging from six months to over 30 years. The term structure of Federal bonds is calculated and published by the Bundesbank on a daily basis.[2] At the beginning of April this year, the German term structure, as calculated by the Bundesbank, was somewhat flatter than its long-term average. The spread between the yields for ten-year and one-year bonds was somewhat more than 0.9 percentage point; on a 30-year long-term average, the spread between the long and short ends of the bond market amounted to 1.26 percentage points.[3]

Shape of the term structure over time

Accordingly, the mean term structure, ie the average of the yield curves over a period of several years, slopes upward. Besides this "normal" shape, which implies an annual yield that increases with the time to maturity of the bonds, the curve may occasionally be inverted. This means that a lower annual yield prevails for longer maturities than for shorter ones; the spread between one-year and ten-year bonds becomes negative. For example, the monetary policy tightening that began in 1979 resulted in short-term interest rates rising to record levels, while the longer-term yields in the capital market did not entirely keep pace: the market players assumed that the increase in short-term rates would be temporary, with rates going back down in the longer term. In line with this, an inverted yield curve could be observed beginning in September 1979. With the decline in interest rates that began in autumn 1981, the interest rate differential gradually returned to "normal" again; from August 1982, the slope of the yield curve was positive. There were similar periods of high short-term interest rates from May 1989 to March 1990, and from Novem-

Term structure of German bond market interest rates*

Current as at 7 April 2006



* For maturities of one, three and six months: money market rates reported by Frankfurt banks. For maturities of one to ten years: interest rates for (hypothetical) zero-coupon bonds (Svensson method), based on listed Federal securities.

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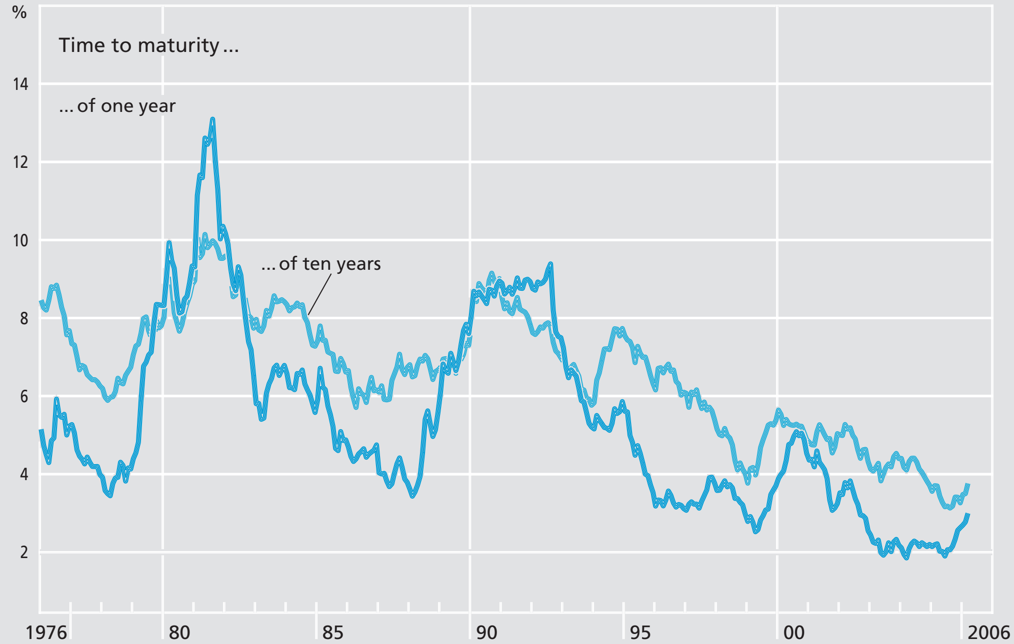
1 In this context, the term structure typically represents the yields of zero-coupon bonds. Such bonds are characterised by the fact that, while no payments are incurred until their maturity, their purchase price is lower than the fixed amount to be paid back. The yield associated with such a bond corresponds to its return, ie the constant annual rate of growth by which the invested capital finally increases up to the amount to be paid out. Unlike zero-coupon bonds, most traded bonds are characterised by the fact that payments (coupons) are paid to the creditor during the life of the bond at pre-determined dates. Nevertheless, in principle, any coupon bond may be expressed as a portfolio of zero-coupon bonds. This means that the price of every coupon bond can be calculated from the term structure of zero-coupon bonds.
2 Using a numerical procedure, the yields on "artificial" zero-coupon bonds are calculated for fixed times to maturity from the bonds traded on the market. A detailed account of the estimation technique and the data used may be found in Deutsche Bundesbank, Estimating the term structure of interest rates, Monthly Report, October 1997, pp 61-66.
3 The average was calculated from the month-end levels from January 1976 to March 2006.

Capital market rates* in Germany

Monthly



* Interest rates for (hypothetical) zero-coupon bonds (Svensson method), based on listed Federal securities.

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ber 1990 to February 1993. The interest rate spread was negative in those periods, too.

Approaches to explaining the shape and dynamics of the term structure

The determinants of interest rates of differing maturities and their behaviour over time are of great interest to financial markets and central banks. For monetary policy, the term structure is of importance in two respects. First, it contains information not only on market expectations of future interest rate movements but also of future developments in inflation and the business cycle. Second, the relationship between short-term and long-term interest rates is relevant to the monetary policy transmission mechanism; although monetary policy has a crucial impact on the short end of the term structure, it is mainly longer-term interest rates which influence decisions on investment, the acquisition of consumer durables or, say, purchasing owner-occupied housing.

Expectations hypothesis of the term structure

The expectations hypothesis is one of the oldest and most prominent approaches to explaining the relationship between interest rates of differing maturities. In its pure form, this hypothesis states that, in equilibrium, investment in a long-term bond is equivalent to the expected return on successive short-term investments. Under this condition, the one-year interest rate, for example, equals the average of the current interest rate and the 11 expected future one-month rates.

Explanatory power and shortcomings

The pure expectations hypothesis thus offers an explanation for the fact that, given low short-term interest rates, the term structure generally slopes upward. If the expectation is that, starting from a very low interest rate level, the short-term interest rates will move towards a more "normal" level again, the long-term rates will accordingly be above those of short maturities. A similar line of reasoning explains why very high short-term interest rates are often accompanied by negatively sloped term structures. The pure expectations hypothesis cannot explain, however, why the term structure is upward-sloping on a long-term average. Indeed, on the basis of this theory, on average a flat term structure would result. An average positive slope of the yield curve would be possible only if short-term interest rates were expected to be rising on average – which is evidently unrealistic.

Term premiums

One explanation for the average positive slope of the yield curve is that investors normally require a "term premium" for a longer-term investment at a fixed rate of interest, which the bond issuer is also willing to pay in order to safeguard longer-term financing conditions.[4] To justify the existence of such a premium, let us look, for example, at an investor who, for an investment horizon of one year, can invest either in a one-year bond or a two-year bond. The return on the one-year bond corresponds to its yield which is known at the time of purchase.[5] If a two-year bond is purchased, however, the investor bears risk because the selling price of this bond in a year's time is unknown. As compensation for this risk, investors who assign a greater weight to the possibility of a capital loss than to a potential capital gain (risk-averse investors) will demand an expected return on the two-year bond that exceeds the one-year interest rate by a premium. Accordingly, term premiums are also referred to as risk premiums.[6]

The expectations hypothesis modified by the existence of term premiums explains the relation between the levels of current short and long-term interest rates by placing the current term structure in relation to the expected movement of future short-term interest rates.[7] The open questions that remain, however, are what determines the absolute level of short-term interest rates, how interest rate expectations are formed and what determines the level and variation of the term premiums over calendar time and time to maturity.

4 For an explanation of the average rising slope of the yield curve, the literature offers various explanatory approaches such as the liquidity premium theory, the preferred habitat theory and the market segmentation theory. See, for example, F S Mishkin (2006), The Economics of Money, Banking and Financial Markets, seventh edition, Pearson, Addison Wesley, or P Bofinger, J Reischle and A Schächter (1996), Geldpolitik, Verlag Vahlen. L Howells and K Bain (2005), The Economics of Money, Banking and Finance, third edition, Prentice Hall, contains a critique of the concept of the liquidity premium.
5 For the sake of simplicity, we consider only bonds for which there is no default risk.
6 In this article, the terms "term premium" and "risk premium" are used synonymously and, unless otherwise specified, denote the differential between the given rate of interest and the notional value that would result on the basis of the pure expectations hypothesis of the term structure. Various definitions of the "term premium" concept may be found in the literature, although some of these are closely related. See, for example, J Cochrane (2001), Asset Pricing, Princeton University Press.
7 There is mixed empirical evidence for the validity of the expectations hypothesis with time-constant term premiums depending on the market observed and the observation period. For an overview, see, for example, B K Cuthbertson and D Nitzsche (2004), Quantitative Financial Economics, second edition, Wiley.

Arbitrage-free models in finance

One avenue of research towards answering these questions comes from the finance literature where the theoretical concept of absence of arbitrage imposes a constraint on the joint movements of short and long-term interest rates. Absence of arbitrage in its strict form means that there is no possibility of achieving a risk-free future profit with a zero net investment.[8] If a trading strategy of this kind were possible – in other words, if the possibility of arbitrage existed – the price adjustments resulting from the trading activities would eliminate the arbitrage opportunity.

Models of monetary macro-economics

Another group of explanatory approaches may be found in the field of empirical macroeconomics.[9] This field investigates the extent to which macroeconomic variables, such as business cycle variables, inflation or exchange rates, determine short and long-term interest rates. Long-term interest rates are explained mostly by assuming the simple expectations hypothesis or – without explicitly considering the relationship between various maturities – by variables such as foreign long-term interest rates, government debt or the volume of household saving.[10]

Linking financial and macroeconomic perspectives

Recent papers seeking to explain the term structure link the approaches of (monetary) macroeconomics with the concept of absence of arbitrage from the finance literature. Before illustrating this combined approach using an example below, the article will discuss, at somewhat greater length, the basic structure of financial models, which do not yet show any explicit relation to macroeconomic models. This basic structure is essentially retained even if macroeconomic aspects are added.

Short-term interest rate as sole explanatory variable

The simplest arbitrage-free models – which are also very prominent in the finance literature – are those in which the short-term interest rate (for example, maturity of one month) itself represents the sole determining component of the whole term structure.[11] The basic component of such a single-factor model is a statistical law of motion which explains the short-term interest rate solely in terms of its own past, with no reference to macroeconomic determinants.[12] At the same time, the statistical description of short-term interest rate movements implies the way in which expectations – in the sense of optimal forecasts – are formed on the basis of currently observed interest rates.

Market price of risk and absence of arbitrage

In this type of single-factor model, the deviations of the short-term interest rates from their expected values represent the only risk

8 A zero net investment is understood as a portfolio of positive and negative shares in bonds of various maturities, with the value of this portfolio being exactly zero. In other words, the value of investment in bonds of one group of maturities is precisely as large as the indebtedness in instruments of other maturities. For a precise definition of arbitrage, see, for example, N H Bingham and R Kiesel (2004), Risk-Neutral Valuation, second edition, Springer, or A Irle (1998), Finanzmathematik, Teubner.
9 A third group of explanatory approaches consists of econometric studies which are solely concerned with the statistical time series characteristics of interest rate processes, especially in the short-term range.
10 See, for example, F A G Den Butter and P W Jansen (2004), An Empirical Analysis of the German Long-Term Interest Rate, Applied Financial Economics, 14, pp 731-741.
11 For more on this approach and the multifactor models presented below, see D Backus, S Foresi and C Telmer (1998), Discrete-Time Models of Bond Pricing, NBER Working Paper No 6736, and Q Dai and K J Singleton (2000), Specification Analysis of Affine Term Structure Models, The Journal of Finance, 55, pp 1943-1978.
12 The interest rate follows what is known as an autoregressive process.

and therefore the sole basis for risk premiums for longer-term bonds: the price which an investor receives after one month for a two-month bond purchased now depends precisely on the prevailing interest rate level in one month's time, which is unknown at present. The size of the additional expected yield compensating for this risk depends on both the fluctuations in the one-month interest rate and the "market price of risk". The latter governs the mark-up or premium which the market "demands" for each additional unit of risk associated with holding a longer-term bond. Finally, the no-arbitrage condition uniquely establishes how the risk premiums are distributed over the maturity spectrum. Put differently, the market price of risk, which can vary over time, determines the general level of the risk premiums at a given point in time, while the no-arbitrage condition fixes their unique cross-section structure.

Linear relationship between short and long-term interest rates

As described above, long-term interest rates are given in the model as the average of expected short-term interest rates and a maturity-dependent and possibly time-varying risk premium. Under certain conditions, this relationship may be represented in an equivalent manner by expressing long-term interest rates as a linear function of the single factor, ie the one-month interest rate.[13] For a given time to maturity, the "slope" of this linear relationship measures the long-term interest rate's reaction to an increase of one unit (0.1 percentage point, for example) in the one-month rate of interest. Slopes and "axis intercepts" differ by maturities and depend, among other factors, on the dynamics of the short-term interest rate, its volatility and the market price of the risk.

Short-term interest rate alone not sufficient to explain term structure

A direct consequence of such a linear relationship between the short-term interest rate and the long-term yields is that interest rates of all maturities have to be completely correlated with one another – in other words, they should co-move perfectly over time. Although interest rate movements across all maturities are indeed highly correlated with each other, this correlation is not perfect. This indicates that the short-term interest rate, as a single determinant, is insufficient as a satisfactory explanation of the joint dynamics of interest rates across the maturity spectrum.

Inclusion of other factors

For this reason, additional "factors" are adduced in most cases. These factors, however, are often not specified in any great detail in the finance literature and are therefore treated as non-observable (latent) variables in empirical studies. In such multifactor models, much as in the single-factor model described above, there are as many sources of risk as there are factors. A market price for the respective risk is assigned to every single one of these factors.

Affine models ...

In the literature, "affine" multifactor models, in which arbitrage-free long-term interest rates can be written as linear combinations of

13 A rate of interest $y(t,n)$ with a maturity of n months at time t depends on the one-month interest rate $i(t) = y(t,1)$, ie as follows: $y(t,n) = A(n) + B(n) \cdot i(t)$, with $A(n)$ and $B(n)$ being variables which depend on time to maturity but are constant over calendar time.

the factors, are especially popular.[14] Apart from its structural simplicity, this representation is also attractive because it generally allows the factors to be interpreted in line with their impact on the differing maturity ranges of the term structure as level, slope or curvature factors.

... explain relative movements of interest rates of differing maturities

Affine multifactor models with latent factors may be used to determine arbitrage-free bond prices over the entire maturity spectrum, to price derivative financial instruments, and for forecasting. These models explain the relative level of interest rates of differing maturities. However, they do not tell us anything about the determinants of the interest rate level itself.

Arbitrage-free term structure and macro-economics

From an economic perspective, however, the macroeconomic factors behind the movement of short and long-term interest rates are of particular interest. A very active recent strand of the literature therefore combines the principle of arbitrage-free valuation with macroeconomic explanatory approaches.[15] This means that the structure of the affine multifactor models outlined above is retained, although some – or all – of the factors no longer remain unspecified but are replaced by concrete macroeconomic variables. These are, for instance, variables such as the inflation rate, the GDP growth rate and other economic indicators or government debt. In line with this, in such models the market prices of risk determine the yield compensation for specific macroeconomic sources of uncertainty (risk of real economic variability, risk of inflation variability etc). As in models with latent factors, here, too, the no-arbitrage condition determines the way in which the interest rates of individual maturities hinge on these macroeconomic variables. This makes it possible, for example, to determine how an interest rate of any maturity will react to an unexpected change in the inflation rate.

Current monetary policy determines interest rates at the short end ...

At the short end of the term structure, the relationship between interest rates and macroeconomic variables is typically interpreted in monetary policy terms. The central bank sets the short-term interest rate in response to inflation, the real economic situation and other relevant macroeconomic variables.

... and long-term interest rates reflect expected economic developments

Long-term interest rates reflect long-term expectations of future macroeconomic developments and risk premiums. These present expectations depend, however, precisely on economic developments up to this time. Accordingly, the current long-term interest rates in affine multifactor models may be repre-

14 In a manner analogous to the explanation in the preceding footnote, an interest rate with a maturity of n periods is given as $y(t,n) = A(n) + B_1(n) \cdot X_1(t) + \ldots + B_d(n) \cdot X_d(t)$, with the variables X_1 to X_d representing the factors. Strictly speaking, the mathematical function $f(x) = a + b \cdot x$ is called linear only if the constant a is equal to zero, otherwise it is called affine.

15 See, for example, A Ang and M Piazzesi (2003), A No-Arbitrage Vector Autoregression of Term Structure Dynamics with Macroeconomic and Latent Variables, Journal of Monetary Economics, 50, pp 745-787; R Fendel (2004), Towards a Joint Characterization of Monetary Policy and the Dynamics of the Term Structure of Interest Rates, Deutsche Bundesbank Research Centre, Discussion Paper Series 1, Economic Studies No 24/2004; G Rudebusch and T Wu (2004), A Macro-Finance Model of the Term Structure, Monetary Policy, and the Economy, Federal Reserve Bank of San Francisco Working Paper 2003-17; P Hördahl, O Tristani and D Vestin (2006), A Joint Econometric Model of Macroeconomic and Term Structure Dynamics, Journal of Econometrics, 131, pp 405-444, and H Dewachter and M. Lyrio (2006), Macro Factors and the Term Structure of Interest Rates, Journal of Money, Credit and Banking, 38, pp 119-140.

sented as a combination of the current macroeconomic variables.

Example: an arbitrage-free term structure model for the development of German interest rates from 1976 up to European monetary union

Basic structure of the model

The way in which such a combined "macro-finance model" works will be illustrated below based on an analysis of the German term structure for the period from January 1976 to December 1998.[16] The model's basic structure may be summarised as follows. Inflation and a business-cycle variable (measured by potential output utilisation, ie the "output gap") are incorporated into the model as macroeconomic variables. Inflation is measured as the deviation of the rate of price increase from the desired rate, expressed by the Bundesbank's "price norm".[17] Furthermore, two other non-observable factors are included in the model. The joint dynamics of inflation and the output gap is described by a vector autoregressive (VAR) model. The part of the model which determines the term structure has the affine structure explained above: arbitrage-free interest rates across all maturities are given as a linear function of inflation, the output gap, and the two non-observable additional factors.[18]

Role of unspecified variables

Leaving some of the explanatory factors unspecified is common practice in the current literature. As a result, the influence on the term structure exerted by numerous additional factors can be captured in condensed form. At the short end,[19] this includes, in particular, additional information variables other than inflation and the output gap which are relevant to monetary policy but have no direct empirical equivalents, such as short-term variations in the "natural" real interest rate, financial system instability, and external factors. At the long end, latent variables reflect fundamentals such as overall productivity.[20]

Applications of such a model

The model which is fitted to the data may be used, for example, to gauge the impact of inflation and cyclical fluctuations on current and future interest rates and to determine the time profile of risk premiums for various maturities. The structuring no-arbitrage condition allows us to derive this information not only for the interest rates employed for esti-

16 Results of similar analyses for the period of European monetary union are not yet very robust as the period since 1999 must be regarded as too short for the econometric estimation methods.
17 For simplicity, this variable will be designated as inflation below. Strictly speaking, by analogy with the output gap, the term "inflation gap" ought to be used. See the annex beginning on page 26 for a precise definition of the variables.
18 The model estimated here essentially follows Ang and Piazzesi (2003). A similar approach may be found in Fendel (2004). See the annex for details of the specification and estimation. Here and in the approaches cited, the joint movement of the output gap and inflation are modelled in a very simple way. The model of Hördahl, Tristani and Vestin (2006), which is likewise estimated for Germany, chooses a more sophisticated approach in a rational-expectations framework.
19 It is in the nature of affine models that all variables affect interest rates across all maturities but that the impact of a given variable varies across maturities. Therefore, if the interpretation of latent variables at the short or long end is the issue, this refers to those maturities where the impact is especially marked.
20 One way of helping to interpret the latent factors might be to compare their estimated paths to those of concrete macroeconomic variables or certain events relevant to interest rates. See, for example, N Cassola and J B Luis (2003), A Two-Factor Model for the German Term Structure of Interest Rates, in Applied Financial Economics, 13, pp 783-806, who choose this procedure in a model with exclusively latent factors.

mating the model[21] but also for interest rates of any given maturity.

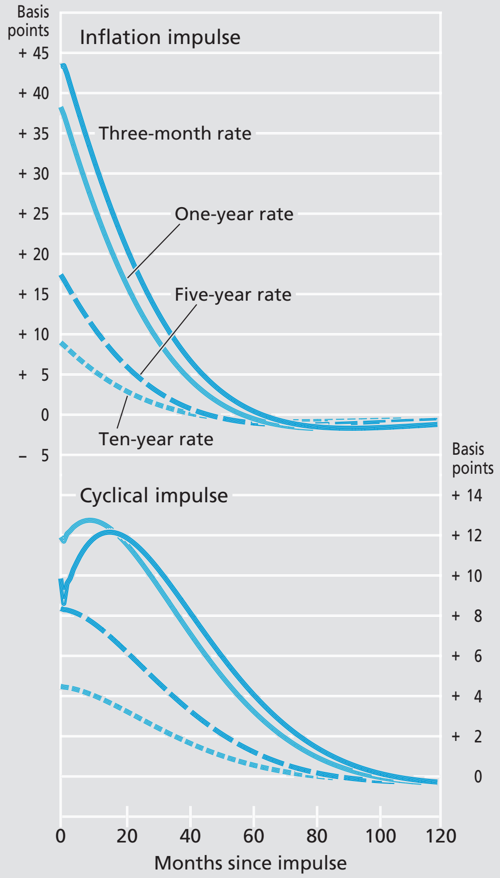
How the term structure responds to an inflation impulse

For a one-off positive inflation shock,[22] let us look, for example, at the effects on interest rates with times to maturity of three months, one year, five years and ten years. When interpreting both this and the subsequent results, it should be borne in mind that alternative models do, of course, produce other quantitative as well as qualitative results. The strongest effect is shown by the three-month interest rate. The response peaks in the period following the impulse. After just under two years, the impact is only half as great as in the first period. The effect on interest rates of longer maturity is likewise positive. Overall, the impact decreases with time to maturity. The original effect of the inflation shock on five and ten-year interest rates is roughly only half and a quarter as high, respectively, as for the one-year interest rate. Furthermore, the impact on longer-term interest rates dies out earlier than that on short-term interest rates.

Transmission of an inflation impulse

From the perspective of the model, the outcome may be explained as follows. The assumed shock increases the inflation rate in the same period and the resulting tighter monetary policy causes the short-term interest rate to rise. Inflation persistence leads to the original effect being reduced only gradually, ie inflation remains above its initial level in the following periods as well. In line with this, all the future short-term interest rates rise as well, albeit to a decreasing extent. At the same time, the assumed inflation impulse can also affect the evolution of the output gap in the following periods. This, in turn, also influences future inflation rates. This leads to further complex effects on the short-term interest rate which either weaken or strengthen the original direct effects. The im-

Interest rate reaction to macroeconomic impulses*



* In the vector autoregressive (VAR) model for inflation and the output gap (both of which are standardised), a one-off, one standard deviation shock of the relevant innovation is generated. An inflation shock of 1 percentage point thus roughly corresponds to 3.15 times the shock documented here, for example.

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21 For the estimation, this article uses interest rates with times to maturity of one month, six months, one year, five years and ten years.
22 In the VAR model for inflation and the output gap (both of which are standardised), a one-off, one standard deviation shock of the relevant innovation is generated. For the derivation of the impulse responses, see Ang and Piazzesi (2003).

Explanatory power of the factors over various time horizons *

Percentages

Factor	Horizon 1 month	12 months	60 months	120 months
	One-month rate			
Inflation	35.77	45.39	46.72	46.37
Output	3.03	3.17	8.77	9.05
Latent 1	20.61	14.49	10.43	10.72
Latent 2	40.59	36.95	34.08	33.86
	One-year rate			
Inflation	39.30	42.70	40.75	40.32
Output	3.83	6.41	12.31	12.39
Latent 1	10.55	6.40	5.77	6.57
Latent 2	46.32	44.48	41.18	40.72
	Five-year rate			
Inflation	22.93	22.23	16.89	16.44
Output	5.25	7.17	9.12	8.88
Latent 1	0.76	5.06	21.47	23.81
Latent 2	71.06	65.53	52.53	50.86
	Ten-year rate			
Inflation	12.58	11.48	7.88	7.63
Output	3.08	3.94	4.49	4.34
Latent 1	9.97	19.64	40.20	42.41
Latent 2	74.36	64.94	47.43	45.62

* Each column contains the percentages of non-forecastable variation (ie deviation from the optimal forecast) of the interest rate in one, 12, 60 or 120 months which are due to variation in inflation, the output gap, the first latent factor and the second latent factor.

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pact on the longer-term interest rates depends, in particular, on the size of the risk parameters and on their sensitivity to inflation and real economic activity.

How the term structure responds to an output impulse

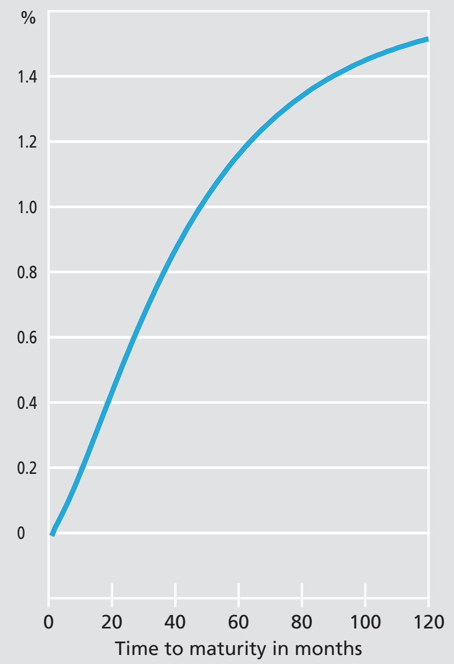
This complex interaction causes the interest rate in the model to respond in a sometimes quite lively manner to macroeconomic impulses. After the initial response of the three-month interest rate to an output shock, the impact on this rate in the following period is somewhat smaller and then finally peters out taking a hump-shaped path, peaking at around 15 months. Over a period of just under 20 months, the impact of the shock on the one-year interest rate is greater than on the three-month interest rate. Over a longer time horizon, the reverse is true. Across all time horizons, the strength of the effect of the shock on five and ten-year interest rates is smaller than for maturities of three months and one year.

Variance decomposition

The explanatory power of the individual factors for the interest rates of various times to maturity can be highlighted by decomposing the theoretical forecast error variance. This states what percentage of the unexpected change in a future interest rate is attributable to innovations of the individual factors for a chosen time horizon (calendar time, not time to maturity).[23] For the one-month and one-year interest rates, the explanatory power of

23 On the basis of the estimated model, it is possible to derive the expected value of the one-year interest rate lying 60 months in the future, for example. Deviations from this forecast are due to the non-anticipatable variations in the four determinants. It is found, for example, that 40.75% of the variability in the one-year interest rate in five years is due to variation in the factor "inflation".

Model-based expected values of the term premiums



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the two explicitly specified macroeconomic factors rises with the increasing time horizon: for the five and ten-year horizons, more than half of the variation in these two interest rates can be explained by fluctuations in the inflation and output variables.[24] For the five and ten-year interest rates, the pattern reverses itself: the explanatory power of the explicitly included macroeconomic variables diminishes as the time horizon increases. Looking at the effects as the time to maturity increases, it can be noted that, for all time horizons, the percentage of the interest rate variation that can be explained by the macroeconomic factors tends to decrease.[25]

Time-varying term premiums and structure of average premiums

The individual term premiums are rather highly time-varying. The long-term average expected risk premiums, as a function of the maturity, show a concave profile. For ten-year instruments, the outcome is a mean term premium of roughly 1½ percentage points.

Summary

Quantitative term structure models are a useful instrument of analysis. They help to understand and quantify the link that exists between short-term interest rates, which can be influenced by monetary policy, and long-term capital market rates, as well as the size and dynamics of risk premiums. This article has illustrated how models from the finance literature are combined with monetary macroeconomics in recent approaches seeking to explain the term structure of interest rates. Essentially, in such approaches, the no-arbitrage condition determines how the impact of macroeconomic variables is spread across interest rates of various maturities.

The example of a simple arbitrage-free multifactor model for the German term structure has been presented to illustrate this approach. From the perspective of such a model, the explanatory power of inflation and economic activity for interest rate movements tends to decrease with the time to maturity. These two macroeconomic factors make the greatest contribution to explaining

24 In qualitative terms, the results are similar to those of Ang and Piazzesi (2003) for the United States. There, however, the macroeconomic factors explain up to 85% of the variance for one-month interest rates for a time horizon of 60 months.
25 Nevertheless, a non-monotone profile can be noted for time horizons of one month and 12 months; for the one-year interest rate, a larger percentage of the variation can be explained by the macro factors than is the case for a time to maturity of one month or five years.

the fluctuations of short-term interest rates in the long term. An analysis of impulse responses shows that the impact of inflationary and cyclical impulses on short-term interest rates is stronger and more persistent than it is on interest rates for longer-term instruments. Furthermore, the model provides evidence that the risk premiums in the observation period were indeed sizeable and also varied considerably.

Annex

A dynamic arbitrage-free model of the term structure – specification and estimation

The joint evolution of the prices of zero-coupon bonds of different maturities is arbitrage-free if a positive stochastic discount factor $M(t)$ exists, such that bond prices satisfy the relationship

(1) $P(t,n) = E_t[P(t+1,n-1)M(t+1)]$

Here, $P(t,n)$ denotes the price of a bond with a time to maturity of n months at time t, and E_t represents expectation based on the information available at time t. Equation (1) restricts the development of bond prices over time and over the various times to maturity. The stochastic discount factor (SDF) in equation (1) is a strictly positive random variable. Modelling the absence of arbitrage using SDF approaches represents a unifying approach to the whole of asset pricing theory.[26]

In microeconomic theory, the form of the SDF can be derived from the optimal investment behaviour of a utility-maximising investor. In this context, the SDF corresponds to the investor's marginal rate of substitution with regard to consumption today and consumption in the subsequent period. However, the literature has shown that using a consumption-based approach for empirical modelling does not yield a satisfactory fit to the observed market interest rates. Generalising the narrow consumption-based approach, then, the SDF is mostly modelled as a function of a set of explanatory variables. The law of motion for these factors, represented by a vector $X(t)$, is formulated here as a first-order vector autoregressive (VAR) process,

(2) $X(t) = KX(t-1) + e(t).$

The SDF depends on these factors and their innovations $e(t)$ in the form

(3) $M(t+1) = exp[-0.5\lambda(t)'\lambda(t) - a - b'X(t) - \lambda(t)'e(t+1)]$

The exponential function is used to ensure the positivity of the SDF and, therefore, the absence of arbitrage opportunities. The vector $\lambda(t)$ includes the market prices of risk: they determine the covariance between the SDF and the impulses on the factors and thus – as can be shown – risk premia, such as the magnitude of excess returns on long-term bonds over the risk-free short-term interest rate. In turn, the market prices of risk are modelled as time-varying and are themselves dependent on the factors $X(t)$ via the parameters d and D,

(4) $\lambda(t) = d + DX(t).$

26 See J Cochrane (2001), loc cit.

Assuming that a zero-coupon bond pays a fixed amount on maturity with certainty, it is possible to calculate the arbitrage-free bond price for any time t and time to maturity n from equation (1) with the help of equations (2) – (4). Finally, by transforming the prices to interest rates[27] using the relationship

(5) $y(t,n) = -(1/n) \cdot log\, P(t,n),$

the arbitrage-free yield $y(t,n)$ of a bond can be expressed as an affine (linear plus a constant) function of the factors,

(6) $y(t,n) = A(n) + B(n)'X(t).$

The constant $A(n)$ and the vector of the factor loadings $B(n)$ are functions of the model parameters, such as the variances of the factors and the risk parameters d and D. The no-arbitrage condition determines the functional form of $A(n)$ and $B(n)$.

In the model outlined in the main article, the term structure is driven by four factors: an inflation variable and an output variable combined in the vector $F^o(t) = (Infl(t), Prod(t))'$, and two unobservable factors, combined in the vector $F^u(t)$. A vector autoregressive model of order p (VAR(p)) is specified for the dynamics of $F^u(t)$[28]

(7) $F^o(t) = Q_1 F^o(t-1) + Q_2 F^o(t-2) + ... +$
$Q_p F^o(t-p) + u(t).$

The latent factors follow a VAR(1) process

(8) $F^u(t) = RF^u(t-1) + v(t).$

The observable macroeconomic factors and the unobservable factors are independent of each other. Collecting $F^o(t)$ and its own lags together with $F^u(t)$ in the vector $X(t)$ allows the factor dynamics (7) and (8) to be represented compactly in equation (2).

In line with Ang and Piazzesi (2003), the estimation takes a two-step approach. Equation (6) shows that, for this specification, the one-month rate can be expressed as

(9) $y(t,1) = a + b_1{}'F^o(t) + b_2{}'F^u(t),$

where b_1 and b_2 are components of the vector b in equation (3). As $F^o(t)$ and $F^u(t)$ are assumed to be independent, it is possible to estimate a and b_1 consistently using an OLS regression of $y(t,1)$ on the inflation and output variables. The VAR(p) of these two variables, equation (7), is also estimated using OLS.[29] The remaining model parameters[30] are determined using a maximum likelihood approach. To do so, the model is converted into the state-space form.[31] It consists of an observation equation in the form

(10) $Y(t) = A + BX(t) + w(t)$

and the factor process (2). The observation vector $Y(t)$ contains five interest rates with differing maturities and the inflation and output variables. The expression $A + BX(t)$ contains the model solu-

27 These and all subsequent interest rates assume continuous compounding.
28 Shocks are identified using a Cholesky decomposition as in Ang and Piazzesi (2003).
29 The lag length is selected using statistical information criteria.
30 These are the variances of the factor innovations, the vector b_2, the risk parameters d and D and the matrix R in the VAR of the latent factors (8).
31 This approach to estimating term structure models is widespread in the literature. See W Lemke (2006), Term Structure Modeling and Estimation in a State Space Framework, Springer Lecture Notes in Economics and Mathematical Systems, Vol 565.

tion. If, for example, the second entry in $Y(t)$ is the six-month rate $y(t,6)$, then the second row of matrix B is given by $B(6)'$ (see equation (6)). The entries in $w(t)$ capture the residuals that are not explained by the model. For the system that results from equations (10) and (2), it is possible to determine the likelihood of the observations using the Kalman filter algorithm. In addition, the path of the latent factors can also be inferred once the parameters have been estimated.

The estimation of the model is based on monthly data from January 1976 to December 1998. The end-of-month levels of Bundesbank-estimated yields from synthetic zero-coupon bonds with maturities of one year, five years and ten years represent one part of our interest rate data. For short-term interest rates, one-month and six-month money market rates are used as reported by Frankfurt banks. Inflation and the output gap are based on the same data used by Hördahl, Tristani and Vestin (2006).[32] To calculate the output gap, they detrend the log of total industrial production (excluding construction) using a quadratic trend. The series is constructed recursively, which means that it only includes data which are available at the point the estimation is conducted. For our model, inflation is calculated as the deviation of the annual rate of change of the monthly CPI from the Bundesbank's "price norm". From 1991 onwards, the data series refer to unified Germany and to western Germany hitherto. Both time series are standardised (ie the mean is subtracted and this value is then divided by the standard deviation) before they are input into the model.

32 Available here: http://www.ecb.int/pub/scientific/wps/date/html/wps2004.en.html.