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A hierarchical Archimedean copula for portfolio credit risk modelling

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Abstract

I introduce a novel, hierarchical model of tail dependent asset returns which can be particularly useful for measuring portfolio credit risk within the structural framework. To allow for a stronger dependence within sub-portfolios than between them, I utilise the concept of nested Archimedean copulas, but modify the nesting procedure to ensure the compatibility of copula generators by construction. This makes sampling straightforward. Moreover, I provide details on a particular specification based on a gamma mixture of powers. This model allows for lower tail dependence, resulting in a more conservative credit risk assessment than a comparable Gaussian model. I illustrate the extent of model risk when calculating VaR or Expected Shortfall for a credit portfolio.

Keywords: Portfolio Credit Risk, Nested Archimedean Copula, Tail Dependence, Hierarchical Dependence Structure

JEL Classification: C46, C63, G21

Non-technical summary

There is a growing consensus among researchers, practitioners and regulators that Gaussian models of asset returns lead to a systematic underestimation of tail risk when applied to a credit portfolio or portfolio credit derivatives. This is due to the fact that such models cannot properly account for the tendency of large negative asset returns to occur simultaneously – the property of the joint distribution of asset returns called in statistical terms lower tail dependence. Therefore, the probability of joint default events and, in turn, extreme portfolio losses is underestimated. It follows that an extensive utilisation of Gaussian models in risk management can be seen as one of the determinants of a massive capital shortfall in the financial sector during the recent crisis. To address this issue, I elaborate a novel model of tail dependent asset returns exploiting the concept of nested Archimedean copulas.

The modelling approach presented in this paper has a number of technical merits which provide it with a beneficial flexibility. I specify two dependence levels – sub-portfolios and the whole portfolio – although the model can be extended to incorporate more than two levels. This hierarchial dependence structure allows for a stronger dependence *within* appropriately specified sub-portfolios (e.g. comprising obligors from a particular industry sector) than *between* them. This is a desirable property because companies in the same sector usually exhibit stronger dependence. The degree of dependence between the companies operating in different industry sectors is, however, lower but not zero because of the influence of a common macroeconomic environment. The model is also flexible enough for differentiated treatment of different sub-portfolios, making it possible to cope with concentration risk. Another advantage is that I avoid by construction built-in parameter restrictions which are otherwise common in the specification of nested Archimedean copulas.

From a computational point of view, the model benefits from a straightforward sampling procedure involving only uniformly and gamma distributed variables. The corresponding pseudo-random number generators are, in fact, standard in mathematical and statistical packages.

Apart from the technical features described, the suggested model could have implications for risk controlling and banking regulation and, on a large scale, for financial stability. Because this model takes account of the tail dependence of asset returns, its implementation would result in a more conservative assessment of portfolio credit risk and, consequently, higher economic and regulatory capital requirements. Therefore, the model is able to counter the systematic underestimation of credit risk in the banking sector – one of the basic causes of the recent financial turmoil.

Nichttechnische Zusammenfassung

Sowohl im akademischen Bereich als auch unter den Fachleuten der Finanzbranche und Regulierern wird vermehrt die Meinung vertreten, dass die auf Normalverteilungsannahmen beruhenden Modelle sich nicht zur Risikomessung von Kreditportfolien oder portfoliobasierten Kreditderivaten eignen. Solche Modelle berücksichtigen ausschließlich lineare Korrelationen zwischen den Firmenwerten der Schuldner. Sie sind nicht in der Lage, die tendenziell gemeinsam auftretenden extremen Firmenwertverluste mehrerer Schuldner zu erklären, die zur Häufung von Kreditausfällen in einem Portfolio führen ("lower tail dependence"). Wenn die Wahrscheinlichkeit gemeinsamer Kreditausfälle unterschätzt wird, so wird auch das Risiko unerwartet hoher Portfolioverluste und der korrespondierende Kapitalbedarf unterschätzt. Daher kann eine verbreitete Anwendung von Kreditportfoliomodellen mit gemeinsam normalverteilten Risikofaktoren als einer der Gründe für die massive Kapitalunterdeckung während der jüngsten Finanzkrise identifiziert werden. Im vorliegenden Papier wird ein stochastisches Modell für Firmenwertrenditen vorgestellt, das die Abhängigkeit der unteren Verteilungsflanken berücksichtigt. Das Modell basiert auf dem Konzept verschachtelter Archimedischer Copulas.

Der hier vorgestellte Ansatz weist eine Reihe technischer Vorteile auf, die dem Modell einerseits Flexibilität verleihen und andererseits seine Umsetzung erleichtern. Ich spezifiziere zwei Abhängigkeitsebenen: die untere Subportfolio-Ebene und die obere Portfolioebene. Diese hierarchische Abhängigkeitsstruktur hat den Vorteil, dass die Firmenwertrenditen von Schuldnern, die ein und demselben Subportfolio zugewiesen sind, stärker voneinander abhängen als die Firmenwertrenditen von Schuldnern aus unterschiedlichen Subportfolien. Wenn die Subportfolien zum Beispiel nach Industriesektoren gebildet werden, so ist für die im gleichen Sektor tätigen Unternehmen die Firmenwertentwicklung hochgradig gleichgerichtet. Dagegen ist die gegenseitige Abhängigkeit von Unternehmen aus verschiedenen Sektoren schwächer, weil sie nur auf das gemeinsame makroökonomische Umfeld zurückzuführen ist. Das Modell ist flexibel genug, um sektorale Unterschiede in der Stärke der Anhängigkeit abzubilden.

Das vorgeschlagene Verschachtelungsprozedere sieht von jeglichen Parameterrestriktionen ab, die sonst bei der Konstruktion hierarchischer Archimedischer Copulas üblich sind. Es gewährleistet darüber hinaus eine einfache Implementierung. So sind für den in diesem Papier näher betrachten Spezialfall nur gamma- und gleichverteilte Zufallszahlen für die Copula-Simulation erforderlich. Die Generatoren für solche Pseudozufallszahlen sind standardmäßig in mathematischen und statistischen Software implementiert.

Von den technischen Vorzügen abgesehen, können aus Anwendung des vorgestellten Portfoliomodells Lehren für die Bereiche Risikocontrolling und Bankenregulierung gezogen werden. Damit einhergehende, konservative Einschätzung unerwarteter Portfolioverluste würde in einem höheren ökonomischen und regulatorischen Kapitalbedarf resultieren, was letztlich einen stabilisierenden Effekt auf das gesamte Finanzsystem haben könnte.

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1. Introduction

Interdependent default events, when defaults of different counterparties tend to occur simultaneously, pose major challenges for an adequate assessment of credit risk in banks' lending portfolios. A prerequisite for estimation of the associated extreme losses is, therefore, a portfolio model capable of capturing dependence between rare events. Under the structural approach for credit risk modelling, a firm's failure results from the asset value of the firm falling below the value of its outstanding debt. Due to this direct link between the default and asset value of a firm, interdependent default events can be modelled based on the joint distribution of asset values or, equivalently, asset returns. Consequently, the tail dependence properties of the joint distribution of asset returns would determine how frequently low-probability events, such as the simultaneous defaults of several obligors, can actually occur. This would eventually affect the amount of portfolio unexpected loss and the capital buffer required as protection against losses.

The above chain of reasoning demonstrates that using portfolio credit risk models with incorporated tail dependence can be crucial for a single bank's ability to remain solvent as well as for the sustainability of an entire banking sector. In spite of that, Gaussian dependence structures have been widely used by practitioners and regulators. Owing to a zero lower tail dependence coefficient, a Gaussian model generates joint defaults far too infrequently, leading to a systematic underestimation of portfolio credit risk and capital requirements and, in turn, endangering banks' solvency.

To address this issue, I put forward a novel model of asset returns which can be used for assessing portfolio credit risk under the structural approach. This model utilises the concept of nested Archimedean copulas.

Archimedean copulas have been actively used in the portfolio risk modelling literature because they possess a simple explicit representation which can be easily extended to an arbitrary number of dimensions. An important feature of various Archimedean copulas is that they allow for tail dependence which does not have to be symmetric. A drawback, however, is the copula's invariance with respect to permutations of its arguments. The latter

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feature is likely to be far too restrictive in a portfolio context, but can be overcome by a nesting procedure, as utilised in this paper.

The nested Archimedean copula described in this paper has two major advantages in terms of portfolio credit risk modelling: the lower tail dependence and the hierarchical dependence structure. The first feature takes account of clustering default events. The latter allows for stronger default dependence *within* appropriately specified sub-portfolios (e.g. comprising obligors from a particular industry sector) than *between* them.[2] It is a desirable property because companies in the same sector usually exhibit stronger dependence. The degree of dependence between the companies operating in different industry sectors is, however, lower but not zero because of the influence of the common macroeconomic environment. The magnitude of dependence parameters can also vary from one sub-portfolio to another so that the model can cope with concentration risk.

Regarding related literature, I refer to Embrechts et al. (2003), McNeil (2008) and Hofert and Scherer (2011) for more information on Archimedean copulas, nesting procedures and compatibility conditions as well as on using nested copulas for credit risk assessment. In the paper at hand, I modify the nesting procedure in a way which ensures by construction the compatibility of copula generators and removes the usual restrictions on copula parameters. From a computational point of view, this redefined nesting procedure makes sampling straightforward. The work presented in this paper was part of my doctoral thesis published in (Puzanova, 2010). In parallel to that work Hering et al. (2010) developed a probabilistic model for construction nested Archimedean copulas using Lévy subordinators. I refer to the latter paper for more general theoretical results on the redefined nesting procedure.

The remainder of the paper is structured as follows: section 2 outlines the credit portfolio setting used in the paper. Section 3 presents the hierarchical Archimedean copula model. In section 4, I apply this model to two hypothetical test portfolios in order to judge the model risk compared with a benchmark Gaussian specification. Finally, I summarise the main results and draw conclusions in section 5.

2. Portfolio setting

I first define target variables which quantify credit risk at the portfolio level. In the tradition of the default-only credit risk models, I look at the distribution of the potential portfolio losses at a one-year risk horizon, whereby the losses can only materialise when one or more borrowers go bankrupt. The portfolio loss, denoted PL, is a random variable which can be defined as the sum over the individual losses L_i on every single exposure in the portfolio (one borrower - one exposure). A loss variable L_i equals zero when the ith borrower survives

[2] I specify two dependence levels – sub-portfolios and the whole portfolio – although the model can be extended to incorporate more than two levels.

beyond the risk horizon. Otherwise, when the borrower defaults on its obligations, L_i equals the expected amount of loss given default LGD_i, i.e., the exposure to this borrower minus the expected recovery. For the sake of convenience, I normalise the maximum total loss to unity so that an individual LGD_i is given as a percentage of the total portfolio LGD.

The event of a borrower i's default is determined in the tradition of structural credit risk models by the standardised returns on the borrower's market value of asset R_i dropping below the default threshold defined by the borrower's one-year probability of default PD_i. The default threshold equals $F_{R_i}^{-1}(PD_i)$, $F_{R_i}^{-1}$ being the quantile function of the random variable R_i. The portfolio setting can be summarised as follows:

$$PL := \sum_{i=1}^{n} L_i = \sum_{i=1}^{n} LGD_i \cdot \mathbb{1}_{\left(-\infty, F_{R_i}^{-1}(PD_i)\right]}(R_i), \tag{2.1}$$

where $\mathbb{1}_A(X)$ denotes an indicator function:

$$\mathbb{1}_A(X) = \begin{cases} 1 & \text{for } X \in A, \\ 0 & \text{otherwise.} \end{cases}$$

With regard to the distribution of the portfolio loss variable PL, I am looking for the Value at Risk at a pre-specified confidence level q (VaR_q) and for the Expected Shortfall (ES_q). VaR is commonly used in risk management and controlling as a measure of portfolio credit risk, although it is incoherent (not sub-additive in general). It quantifies the *minimum* portfolio loss in the worst $(1-q) \times 100$ per cent of cases. $VaR_q(PL)$ equals the value of the quantile function of the random variable PL:

$$VaR_q(PL) := F_{PL}^{-1}(q) \tag{2.2}$$

ES is a coherent risk measure which quantifies the *expected* portfolio loss in the worst $(1-q) \times 100$ per cent of cases. $ES_q(PL)$ equals the conditional tail expectation beyond the q-quantile of the portfolio loss distribution augmented by a discontinuity adjustment:

$$ES_q(PL) := E\big[PL \mid PL \geq VaR_q(PL)\big] + VaR_q(PL) \cdot \frac{1 - q - \Pr\{PL \geq VaR_q(PL)\}}{1-q}. \tag{2.3}$$

3. The hierarchical Archimedean copula model

In this section, I derive a probabilistic model for asset returns utilising nested Archimedean copulas. I also give details on a particular specification based on a two-fold Gamma mixture of powers. I specify the hierarchical dependence structure in subsection 3.1 and investigate its properties in subsection 3.2. Subsequently, I provide an appropriate sampling algorithm in subsection 3.3.

3.1. Derivation of the copula

The portfolio dependence structure should be flexible enough to cope with sub-portfolios which exhibit different dependence properties. Within the scope of Archimedean copulas, this aim can be achieved by utilising a nesting procedure so that the resulting *hierarchcal* Archimedean copula (HAC) would have the following general form

$$C^p\left(C^{sp_1}\left(u_{11},\ldots,u_{1n_1}\right),\ldots,C^{sp_m}\left(u_{m1},\ldots,u_{mn_m}\right)\right).$$

This is a copula model with two hierarchy levels. The copulas C^{sp_j} with $j=1,\ldots,m$ at the lower level of the hierarchy represent dependence functions of distinct groups of obligors. There are m such groups or sub-portfolios each containing n_j obligors with $\sum_{j=1}^m n_j = n$. Those copulas specify the within-group dependence properties. At the top, portfolio level of the hierarchy, the copula C^p joins the sub-portfolios. This copula determines dependence properties between distinct groups of obligors. The copula argument u_{ji} is defined in terms of the probability-integral transform $u_{ji} := F_{R_{ji}}(R_{ji} = x_{ji})$ for the asset return of the obligor i in the sub-portfolio j.

The model should meet the following general requirements. For the purpose of portfolio risk modeling, it is crucial that the copula allows for lower tail dependence. Moreover, the between-group dependence should logically be less strong than the within-group dependence. For the sake of flexibility, the nesting procedure should not pose any technical restrictions on the parameters of the copulas C^{sp_j}.

According to Joe (1997, p. 86 ff.), an Archimedean copula arises from a mixture of powers. The corresponding representation for the joint cumulative distribution function (cdf) of the asset return vector $\mathbf{R}$ is given by:

$$\begin{aligned} F_{\mathbf{R}}(\mathbf{x}) = &\int_0^\infty \int_0^\infty G_{11}^{z_{sp_1}}(x_{11})\cdots G_{1n_1}^{z_{sp_1}}(x_{1n_1})\, dM_{Z_{sp_1}|Z_p}(z_{sp_1} \mid z_p) \times \ldots \\ &\times \int_0^\infty G_{m1}^{z_{sp_m}}(x_{m1})\cdots G_{mn_m}^{z_{sp_m}}(x_{mn_m})\, dM_{Z_{sp_m}|Z_p}(z_{sp_m} \mid z_p)\, dM_{Z_p}(z_p). \end{aligned} \tag{3.1}$$

Z_p is the positive mixing variable at the portfolio level. Its cdf is denoted by M_{Z_p}. Z_{sp_j} represents a sub-portfolio-specific positive mixing variable, which depends by construction on Z_p. The cdf of Z_{sp_j} conditional on Z_p is denoted by $M_{Z_{spm}|Z_p}$.

If, for instance, the sub-portfolios represent the different industry sectors that the borrowing companies are operating in, the economic interpretation of the mixing variables could be as follows: Z_{sp_j} represents all the market and macroeconomic information relevant for the companies operating in sector j. This information is partly sector-specific[3] and not relevant for other sectors. However, in part, it is common for all sectors, such as information about

[3]This part of the information flow will be represented by the random variable Y_{sp_j} later in this text.

the general economic environment. This part of the information is represented by the random variable Z_p.

The two-fold mixture given in (3.1) determines the dependence structure of the asset return vector. The asset returns are mixtures of powers with mixing variables Z_{sp_j}. Thus, conditional on a realisation of Z_{sp_j}, $j = 1, \dots, m$, the returns are mutually independent. The sector-specific random variables Z_{sp_j} are themselves mixtures with the common mixing variable Z_p. Thus, conditional on a realisation of Z_p, the asset returns of the obligors in the different sub-portfolios are independent, but those in the same sub-portfolio are not.

Because in the credit portfolio context only the dependence structure (i.e. copula) of the asset return vector is of interest, we do not have to specify the cdfs G_{ji} exactly. Suffice it to say that, according to Marshall and Olkin (1988), cdfs G_{ji} as implicitly defined by (3.1) always exist.[4]

The HAC corresponding to the mixture representation (3.1) has the form:

$$\begin{aligned} &C^p\Big(C^{sp_1}\Big(u_{11}, \dots, u_{1n_1}\Big), \dots, C^{sp_m}\Big(u_{m1}, \dots, u_{mn_m}\Big)\Big) \\ &= \varphi_p\left[\varphi_p^{-1} \circ \varphi_{sp_1}\left(\varphi_{sp_1}^{-1}(u_{11}) + \dots + \varphi_{sp_1}^{-1}(u_{1n_1})\right) + \dots \right. \\ &\quad \left. + \varphi_p^{-1} \circ \varphi_{sp_m}\left(\varphi_{sp_m}^{-1}(u_{m1}) + \dots + \varphi_{sp_m}^{-1}(u_{mn_m})\right)\right]. \end{aligned} \tag{3.2}$$

Here, φ_p^{-1} serves as the generator of the outer copula C^p and $\varphi_{sp_j}^{-1}$ serves as the generator of an inner copula C^{sp_j}.

Appendix A provides details on the HAC derivation from the general mixture of powers. In this section, I derive a particular HAC model by defining all $M_{(\cdot)}$ in (3.1) as gamma cdfs. This leads to a relatively simple copula function with lower tail dependence. However, it might be an interesting topic for further research to use other positive mixing variables and to compare the dependence properties of the resulting models. Another possible extension is to define more than two hierarchy levels.

Because I specify all mixing distributions in (3.1) as gamma cdfs, I only need to know the Laplace transform (LT) of a gamma random variable in order to derive an explicit formula for the HAC in (3.2). First, I identify the inverse outer generater φ_p as the LT of the gamma variable Z^p with mean 1 and variance κ_p:

$$\varphi_p(\nu) = \big(1 + \nu\kappa_p\big)^{-1/\kappa_p}. \tag{3.3}$$

Second, I identify the inverse inner generator φ_{sp_j} as the LT of the random variable Z_{sp_j}.

[4] These cdfs are given by $G_{ji}(x_{ji}) = \exp\Big\{-\varphi_{sp_j}^{-1}\big[F_{R_{ji}}(x_{ji})\big]\Big\}$, where φ_{sp_j} denotes the Laplace transform of Z_{sp_j}. In general, the Laplace transform of a positive random variable X is given by $\varphi = E\big[\exp\{-\nu X\}\big] = \exp\big[-\Psi\big]$ with Ψ denoting the corresponding Laplace exponent.

Since the conditional distribution of this random variable is the gamma cdf $M_{Z_{spm}|Z_p}$ and the unconditional distribution is a mixture over $M_{Z_{spm}|Z_p}$ with the mixing variable also being gamma-distributed, I refer to the unconditional distribution of Z_{sp_j} as Compound Gamma (CG). It is the distribution of a gamma subordinator, say Y_{sp_j}, evaluated at a random, gamma-distributed time given by Z_p. The gamma random variable Y_{sp_j} has mean 1, variance κ_{sp_j} and LT

$$\varphi_{Y_{sp_j}} = \left(1 + \nu\kappa_{sp_j}\right)^{-1/\kappa_{sp_j}}. \tag{3.4}$$

According to Sato (1999, p. 201), I can derive the LT of Z_{sp_j} as follows:

$$\varphi_{sp_j}(\nu) = \varphi_p\left[-\ln\left(\varphi_{Y_{sp_j}}\right)\right] = \left[1 + \frac{\kappa_p}{\kappa_{sp_j}} \ln\left(1 + \nu\,\kappa_{sp_j}\right)\right]^{-1/\kappa_p}. \tag{3.5}$$

Finally, I check that function (3.2) with φ_p and φ_{sp_j} defined in (3.3) and (3.5) is a proper copula. This is only the case when the compound function

$$w(\nu) := \varphi_p^{-1} \circ \varphi_{sp_j}(\nu) \tag{3.6}$$

satisfies the nesting condition of being in the class $\mathcal{L}_n^*$. This class of functions is specified by Joe (1997, p. 88 f.) as:

$$\mathcal{L}_n^* = \left\{f \;:\; [0,\infty) \to [0,\infty) \mid f(0) = 0,\ f(\infty) = \infty,\ (-1)^{k-1} f^{(k)} \geq 0,\ k = 1,\dots,n\right\}.$$

Because a Lévy subordinator evaluated at a random time given by another Lévy subordinator is also a Lévy subordinator, the copula generators involved are compatible with each other by construction. The compound function $w(\nu)$ is just the Laplace exponent of a Lévy subordinator and, thus, belongs to class $\mathcal{L}_n^*$ (see Hering et al., 2010, Theorem 2.3). This result can also be verified by looking at the derivatives of $w \equiv -\ln\left(\varphi_{Y_{sp_j}}\right)$ which are given by

$$w^{(k)}(\nu) = (-1)^{k-1}(k-1)!\kappa_{sp_j}^{k-1}\left(1 + \nu\kappa_{sp_j}\right)^{-k} \qquad \text{for } k \in \mathbb{N}.$$

The equation above implies that $(-1)^{k-1} w^{(k)} > 0\ \forall \in \mathbb{N}$ holds.

The explicit representation of the HAC, obtained from (3.2) by putting in (3.3) and (3.5), is given by:

$$\begin{aligned} &C^p\left(C^{sp_1}\left(u_{11},\dots,u_{1n_1}\right),\dots,C^{sp_m}\left(u_{m1},\dots,u_{mn_m}\right)\right) \\ &= \left(1 + \kappa_p \sum_{j=1}^{m} \frac{1}{\kappa_{sp_j}} \ln\left[1 - n_j + \sum_{i=1}^{n_j} \exp\left\{\frac{\kappa_{sp_j}}{\kappa_p}\left(u_{ji}^{-\kappa_p} - 1\right)\right\}\right]\right)^{-1/\kappa_p}. \end{aligned} \tag{3.7}$$

This nested portfolio copula produces two interesting special cases of marginal copulas.

The first special case is the n_j-variate Archimedean copula of asset returns assigned to a sub-portfolio j

$$
\begin{aligned}
C^{sp_j}(u_{j1},\dots,u_{jn_j}) &= \varphi_{sp_j}\left[\varphi_{sp_j}^{-1}(u_{j1})+\dots+\varphi_{sp_j}^{-1}(u_{jn_j})\right] \\
&= \left(1+\frac{\kappa_p}{\kappa_{sp_j}}\ \ln\left[1-n_j+\sum_{i=1}^{n_j}\exp\left\{\frac{\kappa_{sp_j}}{\kappa_p}\left(u_{ji}^{-\kappa_p}-1\right)\right\}\right]\right)^{-1/\kappa_p} .
\end{aligned} \tag{3.8}
$$

I term this Archimedean copula compound gamma due to the fact that the inverse generator of the copula is the LT of a CG random variable. The second special case is the copula of asset returns assigned to different sub-portfolios:

$$
\begin{aligned}
C^{p}(u_{1i},\dots,u_{mi}) &= \varphi_{p}\left[\varphi_{p}^{-1}(u_{1i})+\dots+\varphi_{p}^{-1}(u_{mi})\right] \\
&= \left(1-m+\sum_{j=1}^{m}u_{ji}^{-\kappa_p}\right)^{-1/\kappa_p} ,
\end{aligned} \tag{3.9}
$$

which turns out to be a Cook-Johnson copula (or Clayton copula in a two-dimensional case).

It is worth to mention that the nesting procedure put forward in this subsection is different from the commonly used one. For instance, Embrechts et al. (2003) and McNeil (2008) among others suggest, first, to take some known generator functions, nest them as in (3.2) and, second, to proof, whether the inner generators are compatible with the outer generater and, if so, under which parameter restrictions. Apart from the initial uncertainty as to whether the resulting function will be a proper copula or not and the parameter restrictions that must be satisfied, this common procedure makes implementation difficult. In fact, we either do not know the conditional distribution of Z_{sp_j} at all or this distribution is difficult to sample from. By contrast, I use a nesting procedure which, on the one hand, ensures that the compound function (3.6) is actually in $\mathcal{L}_n^*$ without any parameter restrictions. On the other hand, I am free to choose a distribution for the conditional random variable $Z_{sp_j} \mid Z_p$ which is easy to sample from. This feature is especially favorable with regard to the model's implementation. In the particular example elaborated in this subsection the random variable Z_{sp_j} conditional on a realisation $Z_p = z_p$ is gamma-distributed with mean z_p, variance $z_p\kappa_{sp_j}$ and the following LT:

$$
\exp\left\{-z^p\varphi_p^{-1}\circ\varphi_{sp_j}(\nu)\right\} = \varphi_{Y_{sp_j}}^{z^p}(\nu) = \left(1+\nu\kappa_{sp_j}\right)^{-z^p/\kappa_{sp_j}} . \tag{3.10}
$$

3.2. Properties of the copula

Concerning its algebraic properties, the hierarchical Archimedean copula (3.7) is only partially exchangeable in its arguments whereby the marginal copulas (3.8) and (3.9) are fully permutation-symmetric in their arguments.

Regarding their dependence properties, all Archimedean copulas (ordinary or nested) are larger than the independence copula in the sense of concordance ordering.[5] It means that the random variables involved are positively dependent.

Moreover, it is ensured by construction that dependence within sub-portfolios is stronger that dependence between them. According to corollary 4.4.4 in Nelsen (1999, p. 110), the inner copulas are then larger than the outer copula (in the sense of concordance ordering) if the compound function w defined in (3.6) is concave. w is given by $w = -\ln\big(\varphi_{Y_{sp_j}}\big)$ which is the Laplace exponent of the random variable Y_{sp_j} and, thus, a concave function so that the required condition holds.

It follows from the above result that Kendall's tau and other measures of concordance, defined in Nelsen (1999, p. 136f.), are greater for the inner copulas than for the outer copula. It holds:

$$\tau_{\varphi_p} < \min\big\{\tau_{\varphi_{sp_j}}\big\}_{j=1,\dots,m}.$$

The outer Cook-Johnson copula is positively ordered, that is it exhibits stronger dependence for larger parameter values κ_p, as shown in Nelsen (1999, p. 111) for the two-dimensional case. For a fixed value of the parameter κ_p, the inner CG copula is also positively ordered with respect to κ_{sp}. To show that for the two-dimensional case, I use the corollary 4.4.6 in Nelsen (1999, p. 111). According to that corollary, if $\big(\varphi_{sp_1}^{-1}\big)'/\big(\varphi_{sp_2}^{-1}\big)'$ is nondecreasing on $(0,1)$ for a fixed κ_p, then $C^{sp_1} \prec C^{sp_2}$. The derivative of the inverse function of the CG LT is given by

$$\big(\varphi_{sp_j}^{-1}\big)'(\nu) = -\nu^{-\kappa_p-1}\exp\left\{\frac{\kappa_{sp_j}}{\kappa_p}\big(\nu^{-\kappa_p}-1\big)\right\}.$$

It follows that

$$\frac{\big(\varphi_{sp_1}^{-1}\big)'(\nu)}{\big(\varphi_{sp_2}^{-1}\big)'(\nu)} = \exp\left\{\frac{1}{\kappa_p}\big(\nu^{-\kappa_p}-1\big)\right\}^{\kappa_{sp_1}-\kappa_{sp_2}}.$$

This function is nondecreasing for a negative power, that is for $\kappa_{sp_1} < \kappa_{sp_2}$. It follows, that for a fixed value of κ_p the CG family is positively ordered with respect to the parameter κ_{sp}.

To illustrate dependence properties implied by the HAC model and to attain some feeling for the parameter sensitivity, I use information provided by contour and scatter plots for the two-dimensional inner (CG) and outer (Clayton) copulas.

Because of the positive dependence, the contour lines of the copulas under consideration lie between those of the independence copula and those of the maximum copula (the copula of comonotone random variables): cf. Figure 1. Figure 2 illustrates that the magnitude of the positive dependence is the greater, the larger the variance of the mixing variables: the contour lines converge to those of the maximum copula. For a particular value of κ_p, dependence is stronger in the case of the CG copula (i.e. within a sub-portfolio) than for the Clayton

[5] That is, an Archimedean copula is always greater that the independence copula evaluated for the same set of arguments: $C^{Archimedean} \succ C^{\perp}$.

copula (i.e. between sub-portfolios) due to the amplifying effect of the sub-portfolio-specific parameter κ_{sp}.

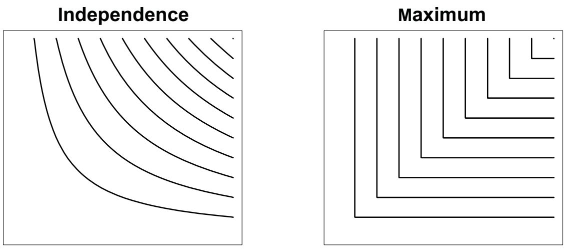
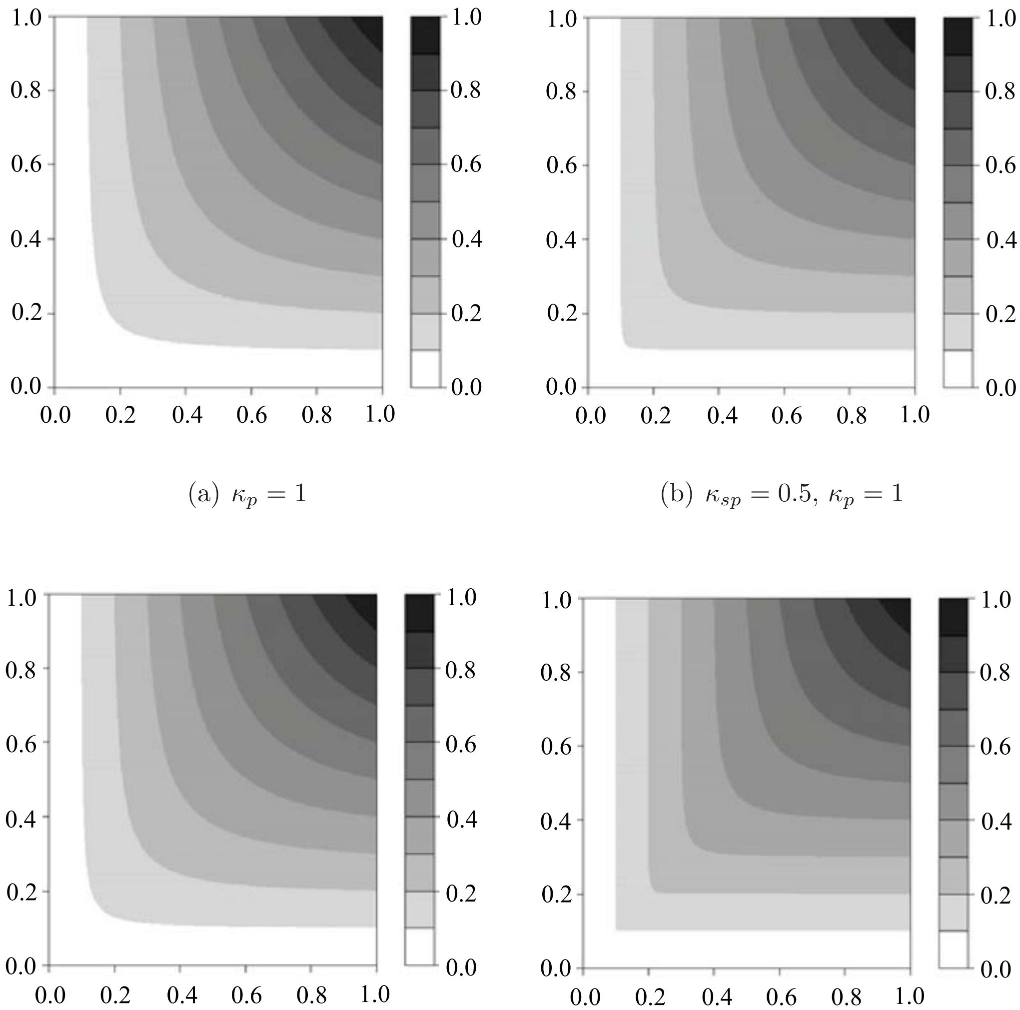


Figure 1: Contour plots of the independence and maximum copulas.



(a) $\kappa_p = 1$

(b) $\kappa_{sp} = 0.5$, $\kappa_p = 1$

(c) $\kappa_p = 1.5$

(d) $\kappa_{sp} = 1$, $\kappa_p = 1.5$

Figure 2: Contour plots of Clayton (left-hand side) and CG (right-hand side) copulas for different parameter values.

With regard to the lower tail dependence, we can clearly see clusters of low-probability realisations in the lower left-hand corners of a unit square in Figure 3. These scatter plots were obtained by simulation using the algorithm provided in the next subsection. The magnitude of lower tail dependence increases with rising values of variance parameters. Again, the sub-portfolio-specific parameter κ_{sp} has an amplifying effect so that the tail dependence within a sub-portfolio is stronger then the tail dependence between asset returns of obligors belonging to different sub-portfolios. A stronger within-sector dependence of the large negative asset returns can also be inferred from the pairwise scatter plots in Figure 4. The figure shows realisations from a four-dimensional Archimedean copula with two hierarchy levels and two sub-portfolios.

The property of lower (upper) tail dependence of two random variables describes the probability that extremely small (large) realisation of both variables occur simultaneously. The general and Archimedean-copula-specific lower and upper tail dependence coefficients are given as (cf. Joe, 1997, p. 103):

$$\lambda_L = \lim_{u \downarrow 0} \frac{C(u,u)}{u} \equiv \lim_{\nu \uparrow \infty} \frac{\varphi(2\nu)}{\varphi(\nu)},$$

$$\lambda_U = \lim_{u \uparrow 1} \frac{1 - 2u + C(u,u)}{1-u} \equiv 2 - 2\lim_{\nu \downarrow 0} \frac{\varphi'(2\nu)}{\varphi'(\nu)}.$$

The lower tail dependence coefficient for the two-dimensional outer (Clayton) copula is given by $\lambda_L = 2^{-1/\kappa_p}$ so that lower tail dependence is increasing in the parameter. For the two-dimensional inner (CG) copula, no explicit expression can be obtained λ_L. But, because of the result $C^p \prec C^{sp}$, it holds that λ_L is greater for the inner copula that for the outer copula (cf. Hering et al., 2010). Moreover, because for a given value of κ_p the CG family is positively ordered with respect to κ_{sp}, the lower tail dependence within a sub-portfolio is stronger for rising values of κ_{sp}.

Regarding the upper tail dependence, it is pointed out by Hering et al. (2010) that if both mixing variables used in the mixture of powers have finite mean, then there is no upper tail dependence between and within sub-portfolios.

I illustrate the parameter sensitivity of the tail dependence by means of a graphic representation in Figure 5. For $u \to 0$ the coefficients converge to values greater than zero indicating positive lower tail dependence. It is greater for the CG copula. For $u \to 1$ the upper tail dependence coefficients converge to zero for both Clayton and CG copulas regardless of the magnitude of the variance parameters.

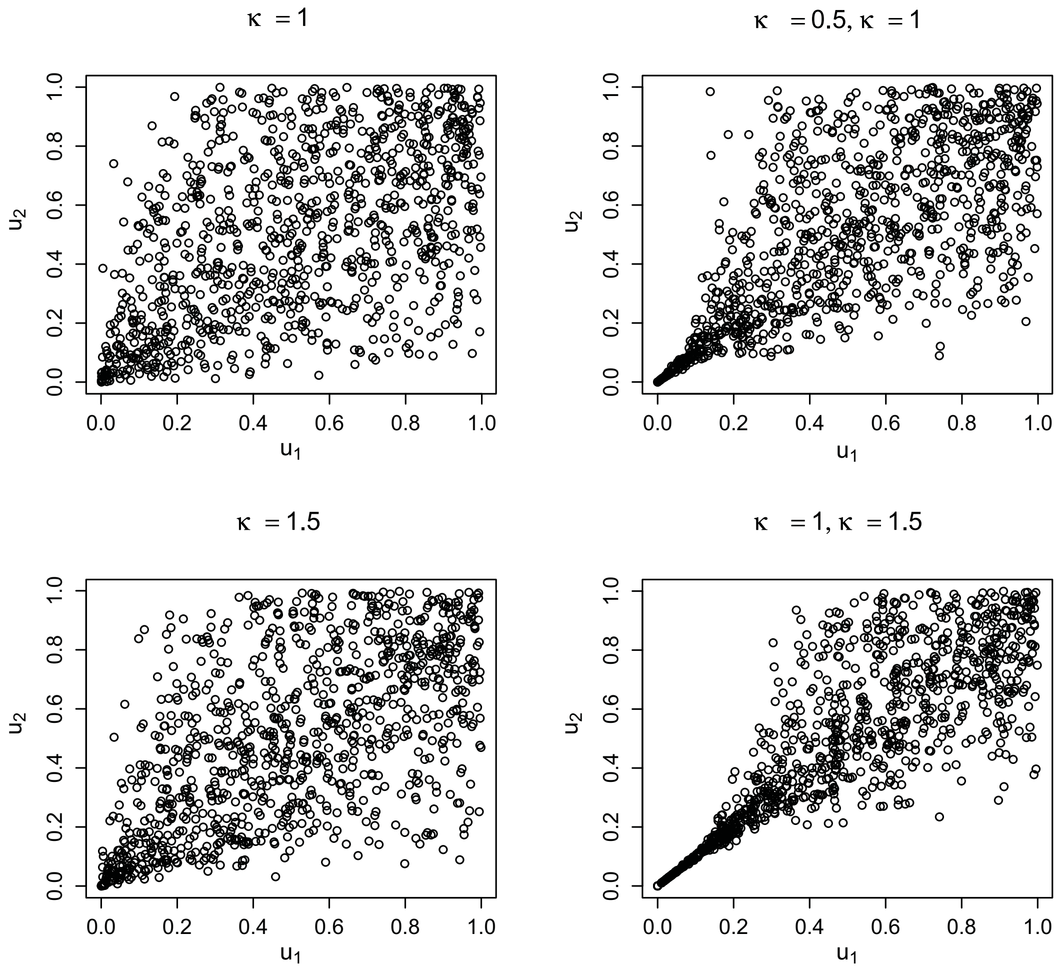
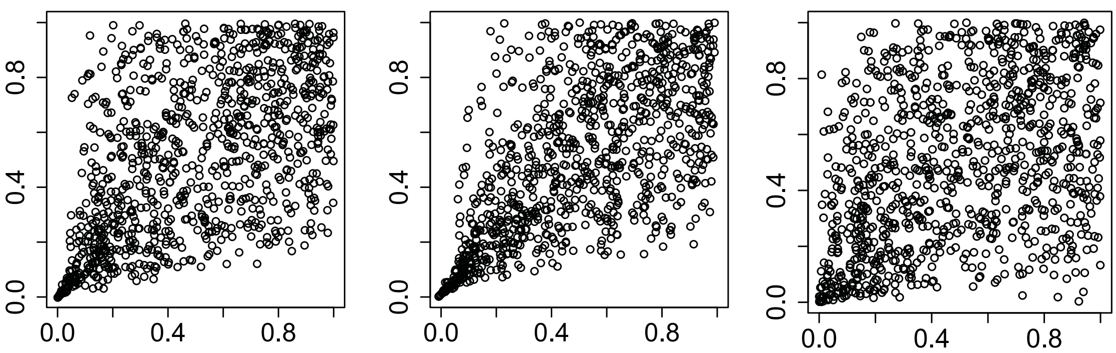


Figure 3: Scatter plots of 1,000 realisations of Clayton (left-hand side) and CG (right-hand side) copulas for different parameter values.



(a) u_{11} and u_{12} (b) u_{21} and u_{22} (c) u_{11} and u_{22}

Figure 4: Pairwise scatter plots of 1,000 realisations of a four-dimensional HAC with two sub-portfolios. Parameter values used are: $\kappa_{sp_1} = 0.25$, $\kappa_{sp_2} = 0.5$ and $\kappa_p = 0.8$.

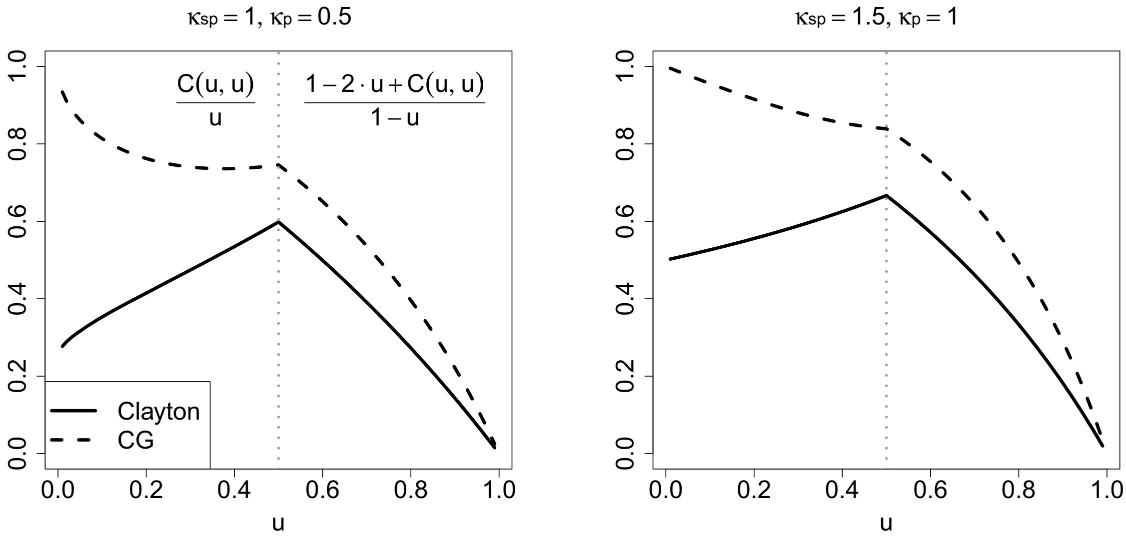


Figure 5: A graphical representation of the tail dependence coefficients for Clayton and Compound Gamma (CG) copulas for different parameterisations.

3.3. Sampling algorithm

I will conclude this section by providing a sampling algorithm for the hierarchical Archimedean copula defined in (3.7). It goes as follows:

- Generate Z_p from the gamma distribution with mean 1 and variance κ_p.
- Generate $Z_{sp_j} \mid Z_p$, $j = 1, \ldots, m$ from the independent gamma distributions with mean Z_p and variance $Z_p \cdot \kappa_{sp_j}$.
- Generate variables X_{ji}, $j = 1, \ldots, m$, $i = 1, \ldots, n_j$ from the uniform distribution on the interval $[0, 1]$.
- The realisations from the HAC are given by
$$U_{ji} = \varphi_p \left\{ -\frac{\ln\Big[\varphi_{Y_{sp_j}}^{Z_p}\Big\{-\ln\big(X_{ji}\big)/(Z_{sp_j} \mid Z_p)\Big\}\Big]}{Z_p} \right\}.$$

The sampling is particularly easy because we know by construction the distribution of the conditional random variable $Z_{sp_j} \mid Z_p$: see (3.10). It was not the case in other papers on nested Archimedean copulas, which I have mentioned in the introductory section.

Having elaborated the theoretical background, I move to an application example in the next section.

4. Application to test portfolios

In order to judge the model risk which arises from neglecting tail dependence in a credit portfolio context, I apply the HAC model introduced in the previous section to two test portfolios and compare the results with the outcomes of a Gaussian model. I describe two test portfolios in subsection 4.1. In subsection 4.2, I calibrate the parameters of the HAC and the benchmark Gaussian model. The final results are discussed in subsection 4.3.

4.1. Test portfolios

The stylised portfolios used in this section are adopt from Puzanova et al. (2009). The smaller portfolio is comprised of 100 credit exposures as summarised in Table 1. 65% of the portfolio LGD is attributed to investment garde (IG) borrowers and 35% to speculative grade (SG) borrowers according to Moody's rating grades/categories. In each rating category, 80% of the total LGD is evenly distributed among 20% of the largest debtors. The remaining 20% of the LGD in each rating category is evenly distributed among the remaining debtors. The second, larger portfolio comprised of 1,000 credit exposures has exactly the same PD-LGD structure as in Table 1 and is obtained from the small portfolio by subdividing each credit exposure into 10 parts.

Table 1: Structure of the small test portfolio

Rating category	PD(%)	Share in the total portfolio LGD(%)	Total # of debtors
IG: Aa	0.064	35	10
IG: A	0.077	15	10
IG: Baa	0.301	15	25
SG: Ba	1.394	15	25
SG: B	4.477	15	25
SG: C	14.692	5	5

Note: Rating categories and corresponding probabilities of default (PD) were obtained from Moody's (2006, p. 33). IG indicates investment grade ratings and SG indicates speculative grade ratings.

4.2. Parameter calibration

In order to assess the model risk evoked by different tail properties of different asset return distributions, I consider a model with a Gaussian correlation structure and calibrate dependence parameters of the HAC model in a way that ensures the same linear correlation.

The obligors in the portfolio are grouped into two sub-portfolios according to their rating grade: sub-portfolio 1 – investment garde (IG), sub-portfolio 2 – speculative grade (SG). I

Table 2: Model parameters used in the simulation

Model	Parameters	Estimates
Gauss	ρ_{sp_1}	0.0321
	ρ_{sp_2}	0.1212
	ρ_p	0.0144
HAC	κ_{sp_1}	0.0214
	κ_{sp_2}	0.1309
	κ_p	0.0175

Note: The correlation parameters used in the simulation for the Gaussian model are set following Puzanova and Siddiqui (2005). The model parameters for the hierarchical Archimedean copula (HAC) are calibrated accordingly in order to maintain the same linear correlation structure.

denote the within-group linear correlation by ρ_{sp_j}, $j = 1, 2$ and the between-group linear correlation by ρ_p. For the purposes of an illustrative example, I take the estimates of the within-group correlation from Puzanova and Siddiqui (2005) and set the between-group correlation to an arbitrarily chosen value which is smaller than the minimum of the within-group correlation coefficients. Numerical values of the correlation coefficients are given in the upper panel of Table 2.

The overall Gaussian copula of asset returns has the correlation matrix given by:

$$\begin{pmatrix} 1 & \cdots & \rho_{sp_1} & \rho_p & \cdots & \rho_p \\ \vdots & & \vdots & \vdots & & \vdots \\ \rho_{sp_1} & \cdots & 1 & \rho_p & \cdots & \rho_p \\ \rho_p & \cdots & \rho_p & 1 & \cdots & \rho_{sp_2} \\ \vdots & & \vdots & \vdots & & \vdots \\ \rho_p & \cdots & \rho_p & \rho_{sp_2} & \cdots & 1 \end{pmatrix}. \tag{4.1}$$

In order to calibrate the dependence parameters of the HAC model, I use the following representation of the Pearson's correlation coefficient:

$$\rho_{X,Y} = \int_0^1 \int_0^1 \left[C_{X,Y}(u,v) - uv\right] dF_X^{-1}(u) dF_Y^{-1}(v) \tag{4.2}$$

for two random variables X and Y.

I put the two-dimensional portfolio-level copula in form (3.9) in combination with Gaussian marginal cdfs onto the right-hand side and the between-group correlation coefficient ρ_p onto the left-hand side of the equation (4.2). Then I solve the equation numerically for the copula parameter κ_p. Subsequently, I substitute the copula on the right-hand side of (4.2) with the bivariate copula in form (3.8) for the sub-portfolio j and the correlation coefficient on the left-hand side with ρ_{sp_j} and solve for κ_{sp_j}. The results are given in the lower panel of Table 2.

4.3. Results of the simulation study

This subsection reports the results of a simulation study for the test portfolios and model settings described previously. In each setting, I run a Monte Carlo simulation for portfolio loss distribution, repeating the simulation loop $s = 1.5 \times 10^7$ times in order to achieve more precise results. I first generate the probability-integral transforms for both models in question.[6] I do this by sampling asset returns from a multivariate normal distribution with the correlation matrix (4.1) and computing $U_i = F_{R_i}(R_i)$ for the Gaussian model. For the HAC model I sample U_i directly according to the algorithm in section 3.3. Then I calculate the default indicators as

$$D_i := \mathbb{1}_{\left(0, PD_i\right]}(U_i). \tag{4.3}$$

For each of the s simulation loops, I calculate portfolio loss as:

$$PL^k = \sum_{i=1}^{n} LGD_i \cdot D_i^k, \quad k = 1, \ldots, s.$$

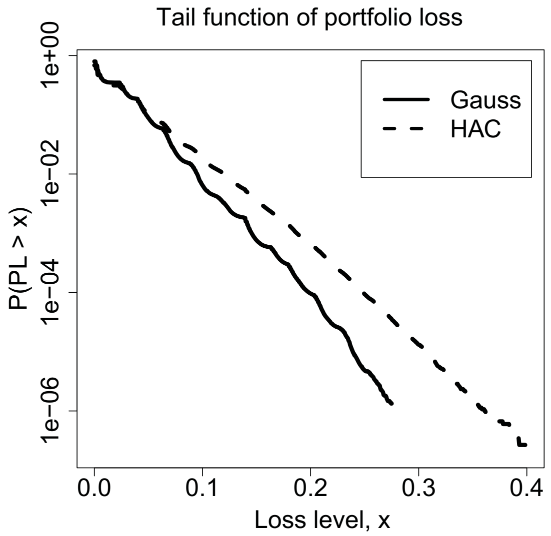
After s simulation runs have been completed, I compute Monte Carlo estimators for the portfolio loss distribution, VaR and ES for each model setting. The estimators are given by:

$$\begin{aligned}
&\hat{F}_{PL}(x_q) \equiv \hat{q} = \frac{1}{s} \sum_{k=1}^{s} \mathbb{1}_{(0, x_q]}(PL^k), \\
&\widehat{VaR}_q(PL) = \inf\left\{x \in [0,1] \, : \, \hat{F}_{PL}(x) \geq q\right\} = PL^s_{\lceil s \cdot q \rceil}, \\
&\widehat{ES}_q(PL) = \frac{\sum_{k=1}^{s} PL^k \, \mathbb{1}_{\left(\widehat{VaR}_q(PL), 1\right]}(PL^k)}{\sum_{k=1}^{s} \mathbb{1}_{\left(\widehat{VaR}_q(PL), 1\right]}(PL^k)} \\
&+ \widehat{VaR}_q(PL) \frac{1 - \hat{q} - \frac{1}{s} \sum_{k=1}^{s} \mathbb{1}_{\left(\widehat{VaR}_q(PL), 1\right]}(PL^k)}{1 - \hat{q}}.
\end{aligned}$$

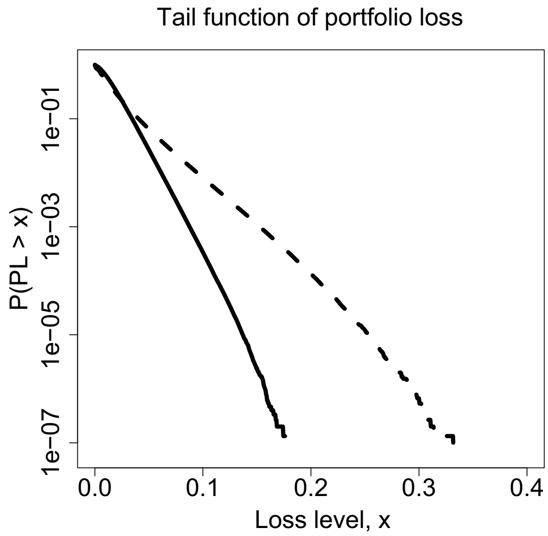
Here, $PL^s_{\lceil s \cdot q \rceil}$ represents the order statistic of the sample $(PL^1, \ldots, PL^s)$ which is either of order $s \cdot q$ or of a larger order next to it.

Figure 4.3 demonstrates that the loss distribution based on the HAC setting has a much more heavier tail than the distribution based on the Gaussian setting, which is in line with the presence of lower tail dependence although the linear correlation is the same. The difference is more pronounced in the case of the larger portfolio because the more debtors are in the

[6] In this representation, the sub-portfolio subscripts are omitted for convenience.



(a) Smaller portfolio



(b) Larger portfolio

Figure 6: Log-lin graphs of the simulated portfolio loss tail function for two different model settings: Gaussian and hierarchical Archimedean copula (HAC).

portfolio, the more combinations of joint defaults are possible and the dependence of rare/tail events has a greater effect.

As to the measures of portfolio tail risk, the simulation results on VaR and ES at different confidence levels are presented in Table 3. In all cases, the tail risk figures within the Gaussian setting are considerably lower than those within the HAC setting. The difference becomes more distinct, the further into the tail we go. Again, the risk of VaR/ES underestimation if the assumption of zero lower tail dependence coefficient is wrong is considerably higher for the larger portfolio, as illustrated in Figure 7. For the same linear correlation of asset returns, the model risk might be very pronounced when true joint distribution of underlying risk factors exhibits lower tail dependence. According to the values at risk marked in the figure for the larger test portfolio, the 99.99%-VaR within the Gaussian setting lies next to the 99%-VaR within the HAC framework for the portfolio. That is, according to the Gaussian model, losses of about 10% of the total portfolio LGD and greater are supposed to occur once every 10,000 years. However, they would occur as often as once a century according to the HAC setting.

The results presented above demonstrate that the tail dependence properties of the underlying joint distribution of asset returns influence to a great extent the tail behaviour of the portfolio loss distribution. Dependence parameters of the HAC model affect the probability of joint borrower defaults and therefore the portfolio tail risk. Table 4 clarifies this issue with an example. Thereby, a simplified model setting is considered with the same sub-portfolio-specific parameters for both IG and SG rating grades ($\kappa_{sp_j} = \kappa_{sp}$). The simulation results for the VaR of the smaller portfolio at different confidence levels demonstrate the impact of the increasing parameter values on the VaR.

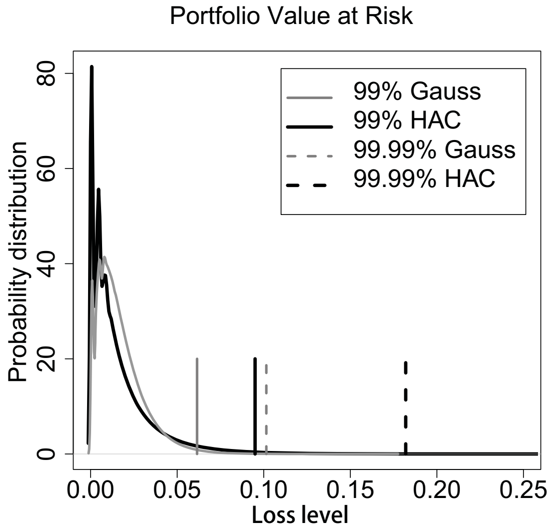


Figure 7: Comparison of the Value at Risk (VaR) of the larger portfolio at two different confidence levels for the Gaussian and hierarchical Archimedean copula (HAC) settings. The probability distributions for both models under consideration are shown and the corresponding VaR values are marked by vertical lines.

5. Conclusions

This paper introduces a novel model for asset returns based on the concept of nested Archimedean copulas. The model takes account of tail dependence and, therefore, can be very useful for the purposes of portfolio credit risk assessment. It exhibits a hierarchical dependence structure which means that it allows for a stronger dependence *within* specified sub-portfolios or sectors than *between* them.

The hierarchical Archimedean copula (HAC) framework presented has a number of technical merits which provide it with a beneficial simplicity and flexibility. Because it is ensured by construction that the resulting function is a proper copula, there is no need any more to check compatibility conditions or parameter restrictions for the nested copula. The sampling from the HAC becomes straightforward.

Apart from the technical features described, the suggested model might have far-reaching implications for risk controlling and banking regulation and, on a large scale, for financial stability. Its implementation would result in a far more conservative assessment of portfolio credit risk and, consequently, higher economic and regulatory capital requirements. Therefore, the model is able to counter the systematic underestimation of credit risk in the banking sector – one of the basic causes of the recent financial turmoil.

Table 3: Comparison of VaR and ES for different settings

q	$\widehat{VaR}_q$ Gauss	$\widehat{VaR}_q$ HAC	$\widehat{ES}_q$ Gauss	$\widehat{ES}_q$ HAC
	Smaller portfolio			
0.9900	0.0955	0.1210	0.1221	0.1514
0.9950	0.1055	0.1415	0.1335	0.1712
0.9990	0.1455	0.1875	0.1634	0.2129
0.9995	0.1665	0.2080	0.1921	0.2330
0.9999	0.1985	0.2485	0.2176	0.2725
	Larger portfolio			
0.9900	0.0615	0.0950	0.0734	0.1214
0.9950	0.0695	0.1125	0.0814	0.1386
0.9990	0.0880	0.1530	0.1010	0.1781
0.9995	0.0960	0.1695	0.1105	0.1930
0.9999	0.1135	0.2065	0.1256	0.2269

Note: VaR and expected shortfall (ES) at different confidence levels q estimated by simulation for various parameter settings and two different models: Gaussian and hierarchical Archimedean copula (HAC). Results are given for two test portfolios containing 100 and 1,000 credit exposures respectively.

Table 4: Parameter sensitivity

	$\widehat{VaR}_{0.99}$			$\widehat{VaR}_{0.999}$		
κ_p	$\kappa_{sp} = 0.2$	$\kappa_{sp} = 0.5$	$\kappa_{sp} = 0.9$	$\kappa_{sp} = 0.2$	$\kappa_{sp} = 0.5$	$\kappa_{sp} = 0.9$
0.01	0.1350	0.1990	0.2540	0.2215	0.3185	0.3490
0.05	0.1535	0.2175	0.2630	0.2735	0.3470	0.3500
0.10	0.1725	0.2345	0.2855	0.3170	0.3500	0.3505

Note: Parameter sensitivity of Value at Risk at different confidence levels (VaR_q) with respect to the dependence parameters of the hierarchical Archimedean copula model. Only the smaller portfolio containing 100 credit exposures is considered.

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A. Derivation of the HAC from the mixture representation

In this appendix I explain in detail the derivation of the HAC given in (3.2) starting from the mixture representation given in (3.1). Beginning with the inner integrals in (3.1), we can write:

$$
\begin{aligned}
&\int_0^\infty G_{j1}^{z_{sp_j}}(x_{j1})\cdots G_{jn_j}^{z_{sp_j}}(x_{jn_j})\,dM_{Z_{sp_j}|Z_p}(z_{sp_j}\mid z_p)\\
&=\int_0^\infty \exp\left\{z_{sp_j}\left[\ln\bigl(G_{j1}(x_{j1})\bigr)+\cdots+\ln\bigl(G_{jn_j}(x_{jn_j})\bigr)\right]\right\}\,dM_{Z_{sp_j}|Z_p}(z_{sp_j}\mid z_p)\\
&=\varphi_{Z_{sp_j}|Z_p}\Bigl[\ln\bigl(-G_{j1}(x_{j1})\bigr)-\cdots-\ln\bigl(G_{jn_j}(x_{jn_j})\bigr)\Bigr] \qquad \text{(A.1)}\\
&=\varphi_{Z_{sp_j}|Z_p}\Bigl[\varphi_{sp_j}^{-1}\bigl(F_{R_{j1}}(x_{j1})\bigr)+\cdots+\varphi_{sp_j}^{-1}\bigl(F_{R_{jn_j}}(x_{jn_j})\bigr)\Bigr], \qquad \text{(A.2)}
\end{aligned}
$$

whereby I use the definition of a Laplace transform (LT) in row (A.1) and the definition of G_{ji} in row (A.2): cf. footnote 4.

$\varphi_{sp_j|p}$ denotes the LT of the random variable Z_{sp_j} conditional on Z_p. It is linked to the LT of the unconditional random variable Z_{sp_j} as follows:

$$
\begin{aligned}
\varphi_{sp_j}(\nu) &= E\bigl[e^{-\nu Z_{sp_j}}\bigr] = E\Bigl[E\bigl[e^{-\nu Z_{sp_j}}\mid Z_p\bigr]\Bigr]\\
&=\int_0^\infty\int_0^\infty e^{-\nu z_{sp_j}}\,dM_{Z_{sp_j}|Z_p}(z_{sp_j}\mid z_p)\,dM_{Z_p}(z_p)\\
&=\int_0^\infty \varphi_{sp_j|p}(\nu\mid z_p)\,dM_{Z_p}(z_p). \qquad \text{(A.3)}
\end{aligned}
$$

When $Z_{sp_j}\mid Z_p$ is an infinitely divisible random variable implying

$$
\varphi_{sp_j|p}(\nu\mid z_p)=\varphi^{z_p}_{sp_j|p=1}(\nu)\equiv\varphi^{z_p}_{Y_{sp_j}}(\nu), \qquad \text{(A.4)}
$$

the expression in row (A.3) can be given as follows:

$$
\begin{aligned}
\varphi_{sp_j}(\nu) &= \int_0^\infty \varphi^{z_p}_{Y_{sp_j}}(\nu)\,dM_{Z_p}(z_p)\\
&=\int_0^\infty \exp\Bigl\{z_p\ln\bigl[\varphi_{Y_{sp_j}}(\nu)\bigr]\Bigr\}\,dM_{Z_p}(z_p)\\
&=\varphi_p\bigl[-\ln\bigl(\varphi_{Y_{sp_j}}\bigr)\bigr]; \qquad \text{(A.5)}
\end{aligned}
$$

cf. (3.5). Solving (A.5) for $\varphi_{Y_{sp_j}}$ results in

$$
\varphi_{Y_{sp_j}}=\exp\bigl\{-\varphi_p^{-1}\circ\varphi_{sp_j}(\nu)\bigr\}; \qquad \text{(A.6)}
$$

cf. (3.10). Combining (A.2), (A.4) and (A.6), the initial mixture of powers representation (3.1) can be rewritten as:

$$F_{\mathbf{R}}(\mathbf{x}) = \int_0^\infty \exp\Bigg\{ -z_p \bigg(\varphi_p^{-1} \circ \varphi_{sp_1} \Big[\varphi_{sp_1}^{-1}\big(F_{R_{11}}(x_{11})\big) + \cdots + \varphi_{sp_1}^{-1}\big(F_{R_{1n_1}}(x_{jn_1})\big) \Big] + \cdots$$

$$+\varphi_p^{-1} \circ \varphi_{sp_m} \Big[\varphi_{sp_m}^{-1}\big(F_{R_{m1}}(x_{m1})\big) + \cdots + \varphi_{sp_m}^{-1}\big(F_{R_{mn_m}}(x_{mn_m})\big) \Big] \bigg) \Bigg\} \, dM_{Z_p}(z_p)$$

$$= \varphi_p \bigg(\varphi_p^{-1} \circ \varphi_{sp_1} \Big[\varphi_{sp_1}^{-1}\big(F_{R_{11}}(x_{11})\big) + \cdots + \varphi_{sp_1}^{-1}\big(F_{R_{1n_1}}(x_{jn_1})\big) \Big] + \cdots$$

$$+\varphi_p^{-1} \circ \varphi_{sp_m} \Big[\varphi_{sp_m}^{-1}\big(F_{R_{m1}}(x_{m1})\big) + \cdots + \varphi_{sp_m}^{-1}\big(F_{R_{mn_m}}(x_{mn_m})\big) \Big] \bigg). \tag{A.7}$$

The representation (A.7) for the joint cdf of asset returns R_{ji} implies the copula (3.2) representation for the corresponding probability-integral transforms $U_{ji} := F_{R_{ji}}(R_{ji})$.

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