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The supervisor's portfolio: the market price risk of German banks from 2001 to 2003 – Analysis and models for risk aggregation

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Discussion Paper
Series 2: Banking and Financial Studies
No 02/2005

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ISBN 3-86558-051-3

Abstract

The Value at Risk of a portfolio differs from the sum of the Values at Risk of the portfolio's components. In this paper, we analyze the problem of how a single economic risk figure for the Value at Risk of a hypothetical portfolio composed of different commercial banks might be obtained for a supervisor. Using the daily profits and losses and the daily Value at Risk figures of twelve German banks for the period from 2001 to 2003, we estimate the Value at Risk of the entire portfolio. We assume a reduced-form model and neglect the effects of a potential bankruptcy of one of the banks. We analyze different models for the cross-correlation of the banks' profits and losses. In an empirical study, we apply backtesting methods to determine which aggregation model leads to the best out-of-sample estimates for the portfolio's economic risk figure. Our main findings can be summarized in three statements. (i) The portfolio's Value at Risk can be estimated from time series data very well. (ii) During 'normal' times, the portfolio's Value at Risk is much lower than the sum of the single Values at Risk. (iii) The relative marginal risk contribution depends on the bank in question and is between 0.05 and 0.62.

Keywords: Value at Risk, portfolio, cross-correlation, market risk regulation, risk forecast, model validation

JEL Codes: C 52, G 11, G 21, G 28

Non-technical summary

The Value at Risk, i.e. the maximum loss that should not be exceeded in a certain period of time with a high probability, is the key figure for measuring market price risks in the banking sector. In this paper, we analyze the question of how a single economic risk figure for the Value at Risk of a hypothetical portfolio composed of different commercial banks might be obtained for a supervisor. We suggest different aggregation models with varying assumptions and check the suitability of these models in an empirical study. This study comprises all of the twelve German banks which used a risk model approved by the German supervisory authorities during the whole of the period under review from 2001 to 2003.

The key results of the empirical study can be summarized as follows.

1. The portfolio's Value at Risk can be estimated from time series data very well.

2. During 'normal' times, the Value at Risk of the portfolio is much lower than the sum of the single Values at Risk.

3. Depending on the bank, the increase in the portfolio's Value at Risk is between 5 and 62 cents for an increase of 1 euro in the Value at Risk of a single bank.

Nichttechnische Zusammenfassung

Der Value at Risk, d.h. der maximale Verlust, der mit hoher Wahrscheinlichkeit innerhalb eines bestimmten Zeitraums nicht überschritten wird, ist eine zentrale Risikokennzahl im Bereich der Marktrisiken im Bankensektor. Im Rahmen dieses Artikels untersuchen wir, wie aus den Risikokennzahlen der einzelnen Banken der Value at Risk eines hypothetischen Portfolios aus Banken berechnet werden kann. Hierzu stellen wir verschiedene Aggregationsmodelle mit unterschiedlichen Annahmen vor. Die Brauchbarkeit der einzelnen Modelle wird anhand unterschiedlicher Validierungsverfahren überprüft. Unsere Untersuchung umfasst alle 12 deutschen Banken, die im gesamten Untersuchungszeitraum von 2001 bis 2003 über ein von der Aufsicht genehmigtes Risikomodell verfügten. Die zentralen Ergebnisse des empirischen Teils unserer Studie sind:

1. Der Value at Risk des Bankenportfolios kann aus den Zeitreihen mit hoher Genauigkeit geschätzt werden.

2. Unter gewöhnlichen Bedingungen liegt der Value at Risk des Bankenportfolios deutlich unter der Summe der Einzel-Values at Risk.

3. Je nach Bank erhöht sich der Value at Risk des Bankenportfolios um 5 bis 62 Cent, wenn der Value at Risk einer einzelnen Bank um 1 Euro erhöht wird.

Content

The Supervisor's Portfolio
The Market Price Risk of German Banks from 2001 to 2003
Analysis and Models for Risk Aggregation[1]

1 Introduction

Value at Risk (VaR) is a key figure for measuring market price risks in the financial sector. Firstly, banks use this risk figure to control their own internal capital allocation. Secondly, Value at Risk can have immediate regulatory effects: after receiving approval from the supervisory authorities, banks are allowed to use their own risk models to calculate the Value at Risk of their trading book in order to determine the regulatory capital cushion.

The relevant literature looks at Value at Risk mainly from two points of view; firstly, from the perspective of a bank that wants to measure and manage the risks of its trading book positions accurately and, secondly, from the perspective of the supervisory authorities which have to deduce the suitability of the banks' risk models from past time series observations (backtesting). In this paper, we choose the perspective of the supervisor. However, our main objective is not to check the suitability of risk models, but instead we aim to measure the risk of a portfolio composed of banks.

We have in mind a fictional supervisor who has to supervise a hypothetical portfolio composed of banks. Our research question is as follows: Assume that the banks in the portfolio keep no capital cushion on their own to cover losses from the trading book, what funds does a lender of last resort need to cover the market price risks of the banking sector? Under the assumption to be confirmed that all the banks' risk models work accurately, the regulator knows the risk of every single bank. However, he does not know the extent of comovement of the profits and losses in the cross-section of the banks or the positions and risk factors for individual banks. Therefore, he cannot aggregate the single risk figures of the banks to obtain the Value at Risk of the entire portfolio. The current – and for good reason established – approach which consists of summing up the Values at

[1]The opinions expressed in this paper are those of the authors and do not need to be those of the Deutsche Bundesbank. We thank Andreas Backes, Frank Heid and Dirk Tasche for their helpful comments.

Risk of each legal entity complies with regulatory concerns about the impact of a single default to the banking system but neglects economic influences such as diversification effects and increasing cross-correlations during times of crises. Therefore, the main topics of this article are (i) to determine the cross-correlation of the profits and losses and (ii) to propose methods of aggregating the Value at Risk figures to obtain a portfolio Value at Risk. This portfolio Value at Risk allows the (fictional) lender of last resort to assess the capital needed to cover the market price risks in the banking sector.

There is a broad range of literature concerning the estimation of market price risks, especially in the area of conditional heteroskedasticity (See Engle (1982), Bollerslev (1986) and Rinne and Specht (2002) for an overview). There is also a large amount of literature on backtesting (for instance Kerkhof and Melenberg (2004)). However, these articles deal mostly with the theoretical foundation of backtesting and seldom with the empirical examination of real data. The lack of empirical studies is most likely due to data constraints. Not surprisingly, most empirical studies in this field are conducted by people in the supervisory environment.

This paper is one of the few studies that resort to real Value at Risk data time series (cf. also Stahl, Traber and Dietz (2002)). We use daily profits and losses and the corresponding VaRs of German banks for the period from 2001 to 2003. The data comprise all of the twelve German banks which used a risk model approved by the supervisory authorities during the whole of the period under review. In addition, the data have the advantage that the daily profits and losses are calculated under the fictional assumption that the banks' trading book positions do not change within the day (clean profits and losses). Other studies in this context have had to make use of the (less suitable) economic profits and losses, for instance Berkowitz (2001) and Berkowitz and O'Brien (2004).

This paper is the first attempt to calculate the Value at Risk of a portfolio of banks from the Values at Risk of single banks. We use several models for the cross-correlation of the profits and losses. When necessary, we estimate correlation parameters from past observations so that the portfolio Values at Risk calculated according to the different models are out-of-sample. We validate these models with the backtesting techniques that are applied to single banks.

Like Jaschke, Stahl and Stehle (2003) and Jaschke, Stahl and Zapp (2004), we observe a relatively low cross-correlation of the profits and losses. This low cross-correlation is the reason why the most appropriate model leads to a portfolio Value at Risk that is only half of the sum of all single Values at Risk. Depending on the bank, the increase in the portfolio's Value at Risk amounts to only between 5 and 62 cents for every 1 euro increase in the Value at Risk of a single bank. Roughly speaking, the regulatory capital for the twelve banks is proportional to the sum of the Values at Risk. As there are diversification effects between the banks, the economic capital requirement is much lower.

As mentioned by Berkowitz and O'Brien (2004), academic literature often warns against destabilizing tendencies due to the systemic adoption of Value at Risk. Our analysis confirms that these warnings are rather of theoretic nature and can be rejected as there are significant diversification effects.

This article is structured as follows. In Section 2, we state our assumptions and give a brief overview of the methods of backtesting. These backtesting methods will later serve as a means of finding the most appropriate aggregation model. In Section 3, we present different approaches to aggregating the risk and calculate marginal risk contributions. Section 4 is about the data description. In Section 5, we present our empirical results. Section 6 concludes this article.

2 Profits and Losses and Value at Risk

2.1 Notation and Assumptions

Let $W_{t,i}^{j}$ be the trading book position of bank *i* of instrument *j=1,...,m* in time *t,* let P_t^j be the price of instrument *j*; then the daily profits and losses of bank *i* can be calculated as

$$G_{t+1,i} = \sum_{j=1}^{m} W_{t,i}^{j}(P_{t+1}^{j} - P_t^{i}). \tag{1}$$

As the positions $W_{t,i}^{j}$ are assumed to be fixed in the course of a day, we refer to $G_{t+1,i}$ as the clean profits and losses (in contrast to the economic profits and losses where price changes and also position changes affect the profits and losses).

We denote the vector of profits and losses for the *n* banks by

$$G_{t+1} = (G_{t+1,1}, \ldots, G_{t+1,n})' . \tag{2}$$

The Value at Risk $V_{t,i}$ is the maximum loss that should not be exceeded with a high probability (here: 99%) in a holding period of one day:[2]

$$\Pr(G_{t+1,i} > V_{t,i}) = 0.99 \qquad \Leftrightarrow \qquad V_{t,i} = F^{-1}_{G_{t+1,i}|W_{t,i}}(1-0.99) \tag{3}$$

The function $F_{G_{t+1,i}|W_{t,i}}(\cdot)$ denotes the cumulative distribution function of the profits and losses $G_{t+1,i}$ of bank *i* conditioned by the holdings $W_{t,i}$. Although the information in the positions $W_{t,i}$ comprises the information in the Value at Risk figures $V_{t,i}$, one can show for the special case of normally distributed price changes that the Value at Risk contains as much information as the positions.

$$F_{G_{t+1,i}|W_{t,i}}(x) = F_{G_{t+1,i}|V_{t,i}}(x) \quad \forall x \tag{4}$$

However, this holds true only if we restrict our analysis to the univariate case of one single bank.

In the following, we assume there to be normally distributed profits and losses. This assumption is justified because we deal with highly aggregated variables; in addition, the on-site inspections show that the normality assumption is a reasonable approximation (See, e.g., Jaschke, Stahl and Stehle (2003)):

$$G_{t+1,i} \mid V_{t,i} \sim N(0, \sigma^2_{t,i}), \qquad \sigma_{t,i} > 0 \tag{5}$$

[2] Without loss of generality, we set the Value at Risk level at 99%. In our definition, the Value at Risk is a negative number.

Moreover, the normality assumption makes the following analysis easier because we have only observations of the form $(G_{t+1,i}, V_{t,i})$ and not predictions or observations of the entire distribution $F_{G_{t+1,i}|V_{t,i}}$. Using the normality assumption, we immediately get

$$V_{t,i} = \Phi^{-1}(0.01) \cdot \sigma_{t,i} \tag{6}$$

which is consistent with "RiskMetrics" by JP Morgan (See Longerstaey (1996)). The term $\Phi^{-1}(0.01)$ is the inverse of the standard normal distribution, calculated at 0.01 (and equals approximately -2.33). In the case of more than one bank, we generalize Equation (5) to

$$G_{t+1} \mid W_t \sim N(0, \Sigma_t) \tag{7}$$

where $\Sigma_t = \left(\sigma_{ij,t}\right)$ denotes a (possibly time-varying) covariance matrix. In contrast to the case for one single bank, the portfolio holdings comprise more information than the Values at Risk. Therefore, it is necessary to condition the profits and losses by the portfolio positions, otherwise one loses information concerning the cross-correlation of the profits and losses. While we have data concerning the banks' Values at Risk, there are no data on their trading book positions. As we cannot observe the cross-correlation, we have to estimate. This is the aim of Subsection 3.1.

The term

$$S_{t+1,i} := \Phi^{-1}(0.01) \cdot \frac{G_{t+1,i}}{V_{t,i}} \tag{8}$$

is the standardized return of bank *i* from time *t* to *t+1*. From Equations (5) and (6), we derive the following for the standardized returns:

$$S_{t+1,i} = \Phi^{-1}(0.01) \cdot \frac{G_{t+1,i}}{V_{t,i}} \sim N(0,1) \qquad iid \tag{9}$$

If we define the covariance matrix Ω_t as $\Omega_t := \mathrm{var}(S_{t+1})$ and the diagonal matrix D_t as $D_t := diag(V_t)$, then we can write the profits and losses as

$$G_{t+1} = D_t S_{t+1} \cdot \frac{1}{\Phi^{-1}(0.01)}. \tag{10}$$

The covariance matrix of the profits and losses is

$$\mathrm{var}\left(G_{t+1} \mid W_t\right) = \frac{D_t{}'\Omega_t D_t}{\left(\Phi^{-1}(0.01)\right)^2}. \tag{11}$$

It can be shown that for each pairwise cross-correlation of the profits and losses the following relation holds:

$$\mathrm{corr}\left(G_{t+1,i}, G_{t+1,j} \mid W_{t,i}, W_{t,j}\right) = \mathrm{corr}\left(S_{t+1,i}, S_{t+1,j} \mid W_{t,i}, W_{t,j}\right). \tag{12}$$

The correlation of the profits and losses is of major importance for the following discussions. Equation (12) states that the correlation of the profits and losses corresponds to the correlation of the standardized returns. In the rest of this paper, we will use the standardized returns (instead of the profits and losses), because the standardized returns have a lot of desirable econometric properties (as can be seen from Equation (9)).

In addition, we define the number of exceedances as $O_{t+1,i} := 1_{(-\infty, V_{t,i})}\left(G_{t+1,i}\right)$. This variable takes the value $O_{t+1,i} = 1$, if the losses of bank *i* exceed the Value at Risk, otherwise it takes the value 0.

2.2 *Validation*

In this subsection, we present some validation methods for risk models. We will use these techniques twice in this paper: (i) to check whether the banks' risk models work accurately and (ii) to find out which of our proposed aggregation approaches leads to the most appropriate estimates for the portfolio Value at Risk.

There are a lot of methods of validating risk models. In this paper, we restrict our attention to the binomial test proposed by BCBS (1996b) and the calibration criteria suggested by Dawid (1982, 1984), Berkowitz (2001), Stahl, Traber and Dietz (2002), Stahl, Wehn and Zapp (2004) .

The binomial test tests the hypothesis that the exceedances occur with a probability of 1%. According to Diebold, Gunther and Tay (1998), the following relation between the random variable and its distribution function exists: Let $\tilde{F}_{G_{t+1,i}|V_{t,i}}$ be the distribution function of $G_{t+1,i}$ conditioned on all past information, then the transformed random variable $Z_{t+1,i} := \tilde{F}_{G_{t+1,i}|V_{t,i}}\left(G_{t+1,i}\right)$ is serially independent and identically uniformly distributed for each bank i.[3] So if the estimated distribution $F_{G_{t+1,i}|V_{t,i}}$ coincides with the actual distribution $\tilde{F}_{G_{t+1,i}|V_{t,i}}$, the random variable $F_{G_{t+1,i}|V_{t,i}}\left(G_{t+1,i}\right)$ is also serially independent and identically uniformly distributed. In the appendix, we make use of this relation and plot the transformed random variable in order to see whether this transformed random variable is really uniformly distributed and serially independent.

Under the normality assumption, the relation from above can be expressed equivalently as:

$$G_{t+1,i} \mid V_{t,i} \sim N(0, V_{t,i}^2 / \Phi^2(0.01)) \quad \forall t \qquad \Rightarrow S_{t+1,i} \sim N(0,1) \qquad iid \tag{13}$$

Validation approaches which analyze the whole estimated distribution $F_{G_{t+1}|V_{t,i}}$ are mainly based on the works of Diebold, Gunther and Tay (1998). In addition, there are approaches from other areas with comparable research questions. For instance, Dawid (1982, 1984) introduces so-called “calibration criteria”. These criteria allow us to analyze whether the transformed random variable (14) differs from realizations of a uniformly and serially independent random variable. Dawid (1982) calls the adequacy of the distributional assumption “*calibration*” and the independence property is often called “*resolution*” (See for instance Overbeck and Stahl (2000)). If there is serial correlation in the transformed random variable, then the dynamics are said to be inappropriately modelled. If $S_{t+1,i} \sim N(0, \sigma_{rec,i}^2)$ with $\sigma_{rec,i} \neq 1$, then the calibration is not perfect and the standard deviation $\sigma_{rec,i}$ is called the “recalibration factor” (See Jaschke, Stahl and Stehle (2003)). In the appendix, we plot the estimated recalibration factors.

[3] This does not imply that the transformed series $Z_{t+1,i}$ is independent in the cross-section.

3 The Risk of the Supervisor's Portfolio

The key contribution of this paper is the aggregation of single Value at Risk figures to obtain the Value at Risk of the entire portfolio. For this aggregation, the covariance matrix of the profits and losses – which is identical to that of the standardized returns (See Equation (12)) – is crucial. We propose different approaches to estimate this covariance matrix. In a second step, we determine marginal risk contributions.

3.1 Estimation of the Covariance Matrix

In this paper, we define the supervisory portfolio as the sum of all the trading book positions across the banks. Therefore, the profits and losses $G_{t,A}$ of this supervisory portfolio are the sum of the profits and losses $G_{t,i}$ of each bank:

$$G_{t+1,A} = \sum_{i=1}^{n} G_{t+1,i} = G_{t+1}'\underline{1} \tag{14}$$

This assumes there to be a free flow of capital between the different parts of the supervisor's portfolio, i.e. between banks. Moreover, we resort to the normality assumption of Equation (5). Due to this assumption, the profits and losses $G_{t,A}$ of the supervisory portfolio are normally distributed as well, because they are a linear combination of normally distributed random variables. The variance of the profits and losses of the supervisor's portfolio amounts to

$$\operatorname{var}\left(G_{t+1,A} \mid W_t\right) = \frac{V_t'\Omega_t V_t}{\left(\Phi^{-1}(0.01)\right)^2}. \tag{15}$$

As – in the case of normality – the Value at Risk is a multiple of the profits and losses standard deviation, we can write the Value at Risk of the supervisor's portfolio as

$$V_{t,A} = -\sqrt{V_t'\Omega_t V_t}\,. \tag{16}$$

Equation (16) shows that the problem of aggregating the Value at Risk figures is reduced to determining the matrix Ω_t. There are various approaches to determining Ω_t. Table 1 shows the approaches which we will analyze in this paper. If there is no recalibration factor (see Subsection 2.2), then the matrix Ω_t will be a correlation matrix, otherwise the main diagonal of Ω_t contains values different from 1 and the matrix is a covariance matrix. In total, we include four assumptions concerning the correlation structure. In each case, we distinguish between whether the recalibration factor equals one or whether it can have any positive value.

Assumption concerning the cross-correlation	Recalibration	
	No recalibration ($\sigma_{rec,i} = 1$)	Empirically recalibrated
Perfectly positively correlated (pc)	*A1*	*A1a*
Uncorrelated (uc)	*A2*	*A2a*
Constant pairwise correlation (cc)	*A3*	*A3a*
Arbitrary correlation structure (ac)	*A4*	*A4a, A4b*

Table 1: Correlation models

In the case of perfectly correlated profits and losses ("pc", cases *A1, A1a*), the Value at Risk of the supervisor's portfolio equals the sum of the single Values at Risk, i.e.

$$V_{t,A1} = \sum\nolimits_{i=1}^{N} V_{t,i} \; . \tag{17}$$

This implies the correlation matrix Ω_t to consist solely of "1". The assumption of perfect correlation is a very conservative approach. The paper by Jaschke, Stahl and Stehle (2003) and our own analyses show that the banks' profits and losses are positively correlated, but that the correlation is relatively weak.

In the case of uncorrelated profits and losses ("uc", cases *A2, A2a*), the correlation matrix Ω_t equals the identity matrix and the portfolio Value at Risk corresponds to the square root of the sum of the squared Values at Risk:

$$V_{t,A1} = -\sqrt{\sum\nolimits_{i=1}^{N} V_{t,i}^{2}} \tag{18}$$

While the assumption of perfect correlation results in an upper bound, the uncorrelated profits and losses may yield the lower bound for the portfolio Value at Risk, because we expect the correlation to be weak, but positive in general.

In the third approach ("cc", cases *A3, A3a*), we allow more flexibility concerning the correlation structure: we assume that all pairwise correlations are equal to ρ_t, which may be time-varying, i.e the correlation matrix Ω_t is given by

$$\Omega_t = \begin{pmatrix} 1 & & \rho_t \\ & \ddots & \\ \rho_t & & 1 \end{pmatrix} \tag{19}$$

The squared portfolio Value at Risk is then the weighted average of the squared portfolio Values at Risk in the cases of perfectly correlated and uncorrelated profits and losses:

$$V_{t,A3}^2 = \rho_t V_{t,A1}^2 + (1-\rho_t) V_{t,A2}^2 , \tag{20}$$

For the empirical application, we replace the true correlation with its empirical counterpart:

$$\hat{\rho}_t := \frac{2}{n(n-1)} \sum_{i=1}^{n} \sum_{j=i+1}^{n} corr(S_{t+1,i}, S_{t+1,j}) \tag{21}$$

If one allows an arbitrary (and possibly time-varying) correlation structure ("ac", cases *A4*, *A4a*, *A4b*), one has to estimate Ω_t from time series data. In our analysis, we estimate the matrix Ω_t from past observations of a rolling window of length *T:*

$$\hat{\Omega}_t = \frac{1}{T-1} \sum_{t'=t-T+1}^{t} \left(S_{t'} - \bar{S}\right)\left(S_{t'} - \bar{S}\right)' . \tag{22}$$

The approaches "pc", "uc" and "cc" force the cross-correlation into a tight structure. The advantage of this tight structure is that only one (or no) unobservable parameter has to be estimated. The fourth approach ("ac") shows much more flexibility with respect to the correlation structure. However, this flexibility comes at a price of having to estimate a large number of parameters (in the case of *n* banks, there are $(n-1)n/2$ pairwise correlations to be estimated and (if applicable) *n* recalibration factors). Therefore, one has to deal with the trade-off between parsimonious, but possibly incorrect models (approaches "pc", "uc" and "cc") and a flexible, but possibly overfitted model ("ac"). We cannot quantify the risk due to a possibly incorrect correlation structure, but we can quantify the estimation risk in the case of model *A4a*. Hence, we modify the case *A4a* to incorporate the estimation error (case *A4b*). The estimation error of case *A4a* can be taken into account as follows: Assuming $S_t \sim N(\mu,\Omega)$, the distribution of the estimated covariance matrix $\hat{\Omega}_t$ (see Equation (22)) is a Wishart distribution:[4]

$$(T-1)\hat{\Omega}_t \sim W(\Omega_t, N, T-1) \tag{23}$$

The distribution of the squared estimated Value at Risk (in the case *A4a*) $V_{t,A4}^2 = V_t\,'\hat{\Omega}_t V_t$ is known as well:

$$T\frac{V_{t,A4}^2}{V_{t,A}^2} \sim W(1,1,T-1) \equiv \chi^2(T-1)\,. \tag{24}$$

Let us define the two stochastically independent processes $X_t := \Phi^{-1}(0.01)\frac{G_{t,A}}{\sqrt{V_{t-1,A}^2}} \sim N(0,1)$ and $Y_t := T\frac{V_{t-1,A4}^2}{V_{t-1,A}^2} \sim \chi^2(T-1)$. These two processes can be used to create a random variable that is Student-*t* distributed (see, e.g., Greene (2000)):

$$\frac{X_t}{\sqrt{\frac{Y_t}{T-1}}} = \Phi^{-1}(0.01)\frac{G_{t,A}}{\sqrt{V_{t-1,A}\,'\hat{\Omega}_{t-1}V_{t-1,A}}} \sim t(T-1) \tag{25}$$

The distribution of Equation (25) depends only on known and observable variables. Therefore, we can explicitly calculate the Value at Risk in consideration of the estimation risk:

$$\Pr(G_{t,A} < V_{t-1,A4b}) = 0.01 \;\; with \qquad V_{t,A4b} = \frac{t_{0.01,T-1}}{\Phi^{-1}(0.01)}V_{t,A4a}\,, \tag{26}$$

where $t_{0.01,T-1}$ denotes the 1%-quantile of a Student-*t* distribution with *T-1* degrees of freedom.

[4] The Wishart distribution can be seen as the multivariate extension to the χ^2 distribution. See, e.g., Anderson (1984) und Press (1972).

The estimator $V_{t,A4}^2$ is an unbiased estimator of the squared Value at Risk $V_{t,A}^2$ in case we know the correlation matrix Ω_t. However, the estimated portfolio Value at Risk $V_{t,A4b}$, which takes the estimation error into account, is always greater than this unbiased estimator $V_{t,A4a}$. If one uses T=50 observations for the estimation, the modified Value at Risk $V_{t,A4b}$ is by roughly 3.4% higher than the unbiased estimator $V_{t,A4a}$.

3.2 Marginal Risk Contribution

Only in the implausible special case of perfectly correlated profits and losses does the portfolio Value at Risk increase by the same amount if the Value at Risk of a single bank is increased. As a rule, the increase in the portfolio Value at Risk is significantly lower than the increase in the Value at Risk of a single bank. In this subsection, we aim to determine the marginal risk contribution of the banks to the risk of the supervisor's portfolio.

The marginal risk contribution of bank i is defined as

$$V_{t,i_0}^{\partial} := \frac{\partial}{\partial h} V_{t,A}\left(G_{t+1,A} + hG_{t+1,i_0}\right)\Big|_{h=0}. \tag{27}$$

In the case of normally distributed profits and losses, it is possible to express the marginal risk contribution in a closed form. This expression for the marginal risk contribution is reminiscent of the risk contribution in the Capital Asset Pricing Model (CAPM):[5]

$$V_{t,i_0}^{\partial} = \Phi^{-1}(0.01)\frac{\operatorname{cov}\left(G_{t+1,A}, G_{t+1,i_0}\right)}{\sqrt{\operatorname{var}\left(G_{t+1,A}\right)}} \tag{28}$$

For the relative marginal risk contributions we get:

$$\begin{aligned} \frac{V_{t,i_0}^{\partial}}{V_{t,i_0}} &= \Phi^{-1}(0.01)\frac{V_{t,A}V_{t,i_0}\sqrt{\Phi^{-1}(0.01)^2}\operatorname{cov}\left(S_{t+1,A}, S_{t+1,i_0}\right)}{\Phi^{-1}(0.01)^2\sqrt{V_{t,A}^2\operatorname{var}\left(S_{t+1,A}\right)}V_{t,i_0}} \\ &= \frac{\operatorname{cov}\left(S_{t+1,A}, S_{t+1,i_0}\right)}{\sqrt{\operatorname{var}\left(S_{t+1,A}\right)}} \end{aligned} \tag{29}$$

[5] See Sharpe (1964), Lintner (1965) and Mossin (1966).

The relative marginal risk contribution (Equation (29)) shows how much the portfolio Value at Risk increases if the Value at Risk of a single bank rises by one euro.

4 The Data

The data we use consist of the pairwise observations $(G_{t+1,i}, V_{t,i})_{t=1,\ldots,T}$ of twelve German banks. This group of banks includes all the German banks with a risk model for their trading book approved by the supervisory authorities and in use from 2001 to 2003. $G_{t,i}$ are the daily profits and losses of bank *i*, where these profits are calculated under the assumption that the bank's portfolio composition does not change in the course of the day (clean profits and losses). $V_{t,i}$ is the corresponding Value at Risk for the period of one day and a level of 99%.

4.1 Descriptive Statistics

In Table 2 we report the first four moments of the distribution of the standardized returns $S_{t,i}$. The mean of this distribution is always close to zero, the standard deviation is often marginally less than one. The skewness differs only slightly from zero, so that we can maintain the assumption of a symmetric distribution. However, the kurtosis often exceeds the value – the normal distribution has a kurtosis of 3 – that is in accordance with the normal distribution. Nevertheless, the QQ-plots as a measure of goodness-of-fit (as depicted in Figure 2 in the appendix) show that the normality assumption can be seen as a suitable approximation. The calibration and resolution criteria can be seen as being largely fulfilled.

Bank	Standardized Returns Mean	Standard Dev.	Skewness	Kurtosis
A	0.0563	1.2047	0.6953	7.2934
B	0.1496	0.8991	0.0000	4.7137
C	0.1076	0.8644	0.1528	8.9625
D	0.0081	0.9889	0.0227	3.1592
E	0.0034	0.9439	0.1411	5.2921
F	0.2894	0.8645	-0.0412	4.7519
G	0.0375	0.7079	-0.2713	5.7379
H	0.0428	0.5899	-0.2074	5.3618
I	0.0179	1.1559	-0.4062	6.0821
J	0.0565	0.8581	-0.1092	3.4968
K	-0.1311	0.7722	-0.1698	5.3099
L	0.0879	0.5888	0.0583	3.7686

Table 2 Descriptive Statistics

For reasons of anonymity, we cannot report summary statistics concerning the Value at Risk exceedances $O_{t,i}$. However, we can say that the exceedances are not clustered in the cross-section. Even the time around September, 11th 2001 was not characterized by an accumulation of exceedances.

4.2 Cross-Correlation of Profits and Losses

The cross-correlations of the profits and losses are essential for the risk aggregation. Therefore, we will report examples of these correlations and show their development over time. In Table 3, we show the estimated correlation matrix for the point in time *t*=245. The aim is to give an impression of the magnitude and diversity of the cross-correlation. On average, the correlation is low and can even be negative. There are remarkable patterns like the negative correlation of bank A with respect to all other banks.

	A	B	C	D	E	F	G	H	I	J	K
B	-0.31										
C	-0.15	0.09									
D	-0.07	0.29	-0.01								
E	0.06	-0.22	0.15	-0.16							
F	-0.33	0.26	0.06	0.55	0.07						
G	-0.26	0.38	0.20	0.50	0.04	0.55					
H	-0.23	0.35	0.22	0.31	0.15	0.47	0.74				
I	-0.38	0.42	0.09	0.49	0.08	0.78	0.71	0.65			
J	-0.23	0.27	0.10	0.19	-0.17	0.09	0.22	0.16	0.17		
K	-0.03	0.18	-0.05	0.23	0.08	0.15	0.26	0.26	0.21	0.27	
L	-0.42	0.20	0.09	0.37	0.11	0.57	0.49	0.52	0.73	0.15	0.01

Table 3: Correlation matrix for t= 245.

Figure 1: Mean cross-correlation ρ_t in the course of time

In Figure 1, we plot the mean cross-correlation ρ_t as a function of time. The average mean correlation is 0.05. We see peaks following the events of September, 11th 2001.

5 Empirical Study

5.1 Preliminary Remarks

For the following analysis (Subsection 5.2) we presume that all Values at Risk reported by the banks are accurate. Moreover, we assume that the standardized returns are a standard normally distributed and serially independent time series. We checked these assumptions in univariate analyses of the single banks. The time series of the standardized returns and their squares show only minor serial correlation and the fraction of the explained variance that is due to past observations, R^2, is insignificant. These results are an indication of efficient markets and the good quality of the banks' risk forecasts, i.e. there is no systematic underassessment or overassessment of the risk. The assumption of normality proves to be a good approximation as well. By way of exemplary evidence, we report the QQ-plot of three banks in Figure 2 in the appendix.

Generally speaking, we can say that the banks adequately measure their market risks. Therefore, we can maintain the above assumption that the single Values at Risk are accurate and the main aggregation task consists of determining the correct correlation structure.

5.2 Comparison of Different Aggregation Approaches

In this subsection, we apply the different aggregation approaches of Subsection 3.1 to the data described in Section 4. We take the position of a regulator who knows the Value at Risk figures of all the banks for the following trading day. This regulator estimates the unobservable parameters necessary for the risk aggregation, i.e. correlations and recalibration factors, out of the last T=50 observations. It is important to state that our hypothetical regulator uses only past and no future information. Therefore, all of the estimated portfolio Values at Risk can be used for out-of-sample risk forecasts.

Table 4 displays descriptive statistics for the different aggregation approaches. We report the number of observations, which start in t=51 and go to t=731, the number of exceedances $O_{t,i}$, the mean and standard deviation of the standardized returns $S_{t,i}$, and the average Value at Risk as a percentage of the average sum of the single Values at Risk (which is identical to the average Value at Risk of case *A1*).

As is to be expected, the assumption of perfect correlation ("pc", cases *A1* and *A1a*) leads to strongly conservative Values at Risk. If we assume uncorrelated profits and losses ("uc", cases *A2* and *A2a*) instead, the portfolio Value at Risk will be too aggressive and the QQ-plot in the appendix shows only little fit. In our view, the approaches *A4b* and *A3a* prove to be the best aggregation methods. The backtesting methods for these two aggregation approaches reveal no shortcomings, for instance, no clustered exceedances and little serial correlation in the squared standardized returns. The calibration and resolution criteria are best met within the latter two approaches.

Correlation model		Number of Obs.	Number of Exceed.	Stand. Returns Mean	Stand. Returns Std. Dev.	Average VaR A1=100%
pc	*A1*	681	0	0.060	0.443	100%
	A1a	681	0	0.082	0.549	79%
uc	*A2*	681	9	0.123	0.914	49%
	A2a	681	19	0.177	1.175	38%
cc	*A3*	681	5	0.118	0.825	53%
	A3a	681	7	0.139	0.933	47%
ac	*A4*	681	3	0.115	0.793	56%
	A4a	681	11	0.157	1.036	44%
	A4b	681	10	0.152	1.002	45%

Table 4: Descriptive statistics for the different aggregation approaches

To sum up: we have found two aggregation strategies that allow us to appropriately calculate the portfolio Value at Risk out of the single Values at Risk of the portfolio components. The average Values at Risk of these two approaches (*A3a, A4b*) is less than half of the average sum of the Values at Risk (*A1*). This means that the supervisor's portfolio benefits from diversification effects.

On the basis of Approach *A4b,* we estimate the relative marginal risk contributions according to Equation (31). Depending on the bank, the estimated risk contributions range from 0.052 to 0.624. Larger banks tend to have greater relative risk contributions. That is why the unweighted average risk contribution is 0.244 instead of around 0.5.

6 Conclusion

This paper is the first attempt to calculate the Value at Risk of the supervisor's portfolio composed of banks. We suggest several aggregation approaches, two of which satisfy our backtesting requirements: one that assumes constant pairwise correlation and one that uses the full covariance matrix. Applying these two approaches, we can reduce the average portfolio Value at Risk to half of the sum of all single Values at Risk. This reduction confirms the finding that the cross-correlation of the banks' profits and losses is low. The marginal relative risk contributions depend on the bank in question and range from 5 to 62 cents per 1 euro.

From a systemic point of view, the good diversification of the supervisor's portfolio is comforting. However, the Value at Risk is a measure for normal situations. The cross-correlations may become quite important in times of financial crisis. At such times, the diversification benefits will diminish. It should be mentioned that, owing to the cascading effects caused by the bankruptcy of one single bank (see e.g. Jorion (2001)), the assumption of free cash flow is for good reason not shared by the regulation that aims to keep the financial system stable.

The data set used in this paper can help to answer a lot more questions. Possible research questions are the identification of common risk factors (stock markets, interest rates, exchange rates) and the analysis of the Values at Risk in the course of time, especially during periods of stress.

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8 Appendix: Validation and Visualization

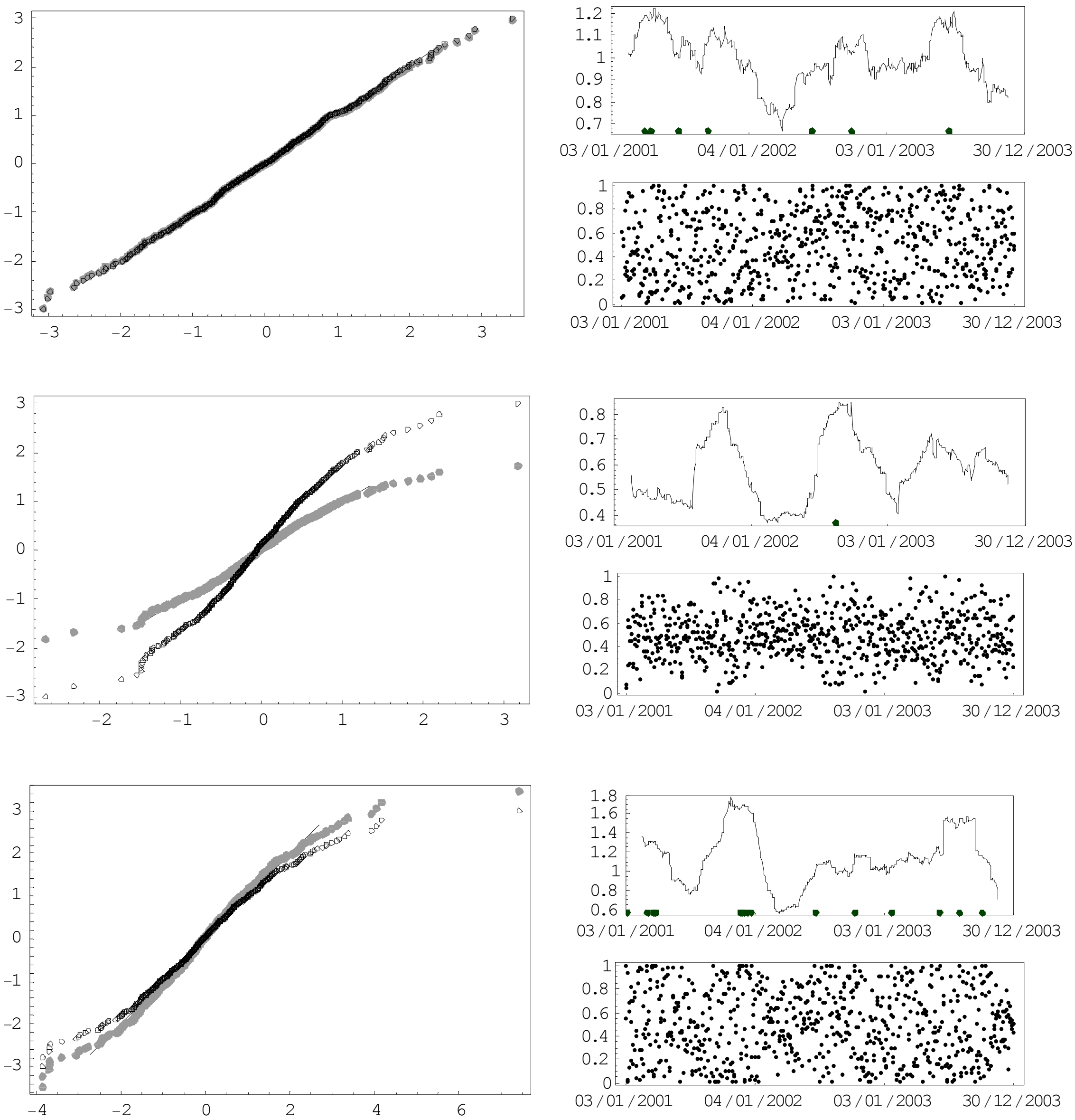

Figure 2: Exemplary validation of three arbitrary banks: QQ-plot of standardized returns S_{t+1} (with (lighter)/without (dark) recalibration $\sigma_{rec,i}$, based on a moving window), estimated recalibration factor with exceedances, and transformed profits and losses Z_{t+1}.

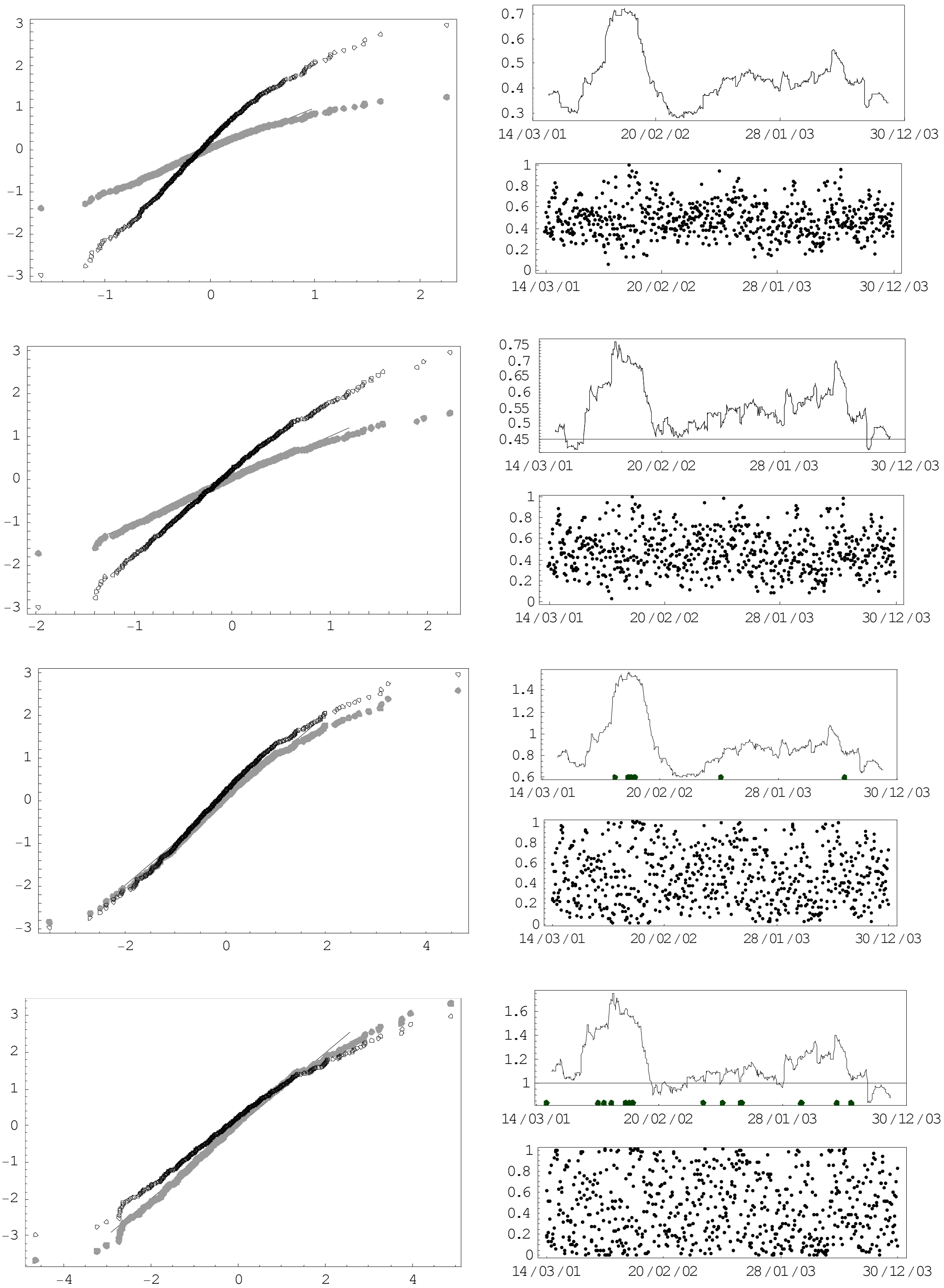

Figure 3: Validation of the aggregation approaches A1, A1a, A2 and A2a ("pc" and "uc", from top to down): QQ-plot of standardized returns S_{t+1} (with (lighter) / without (dark) recalibration $\sigma_{rec,i}$, based on a moving window), estimated recalibration factor with exceedances, and transformed profits and losses Z_{t+1}.

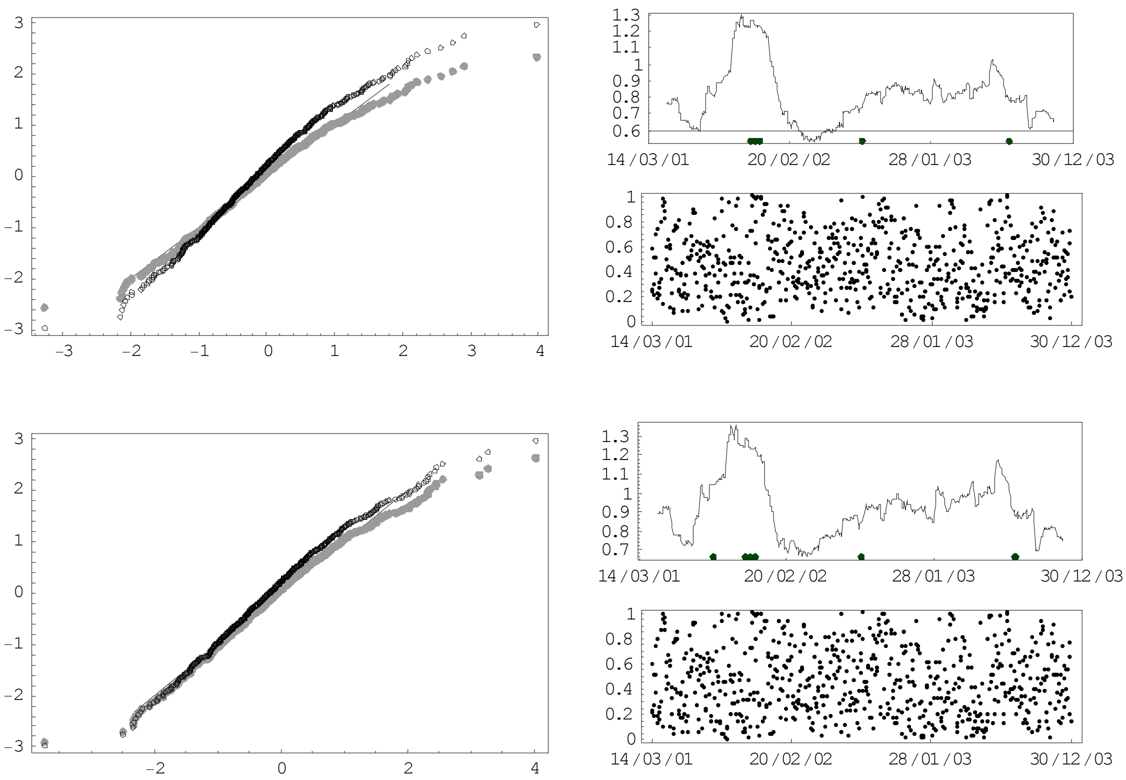

Figure 4: Validation of the aggregation approaches A3 and A3a ("cc", from top to down): QQ-plot of standardized returns S_{t+1} (with (lighter) / without (dark) recalibration $\sigma_{rec,i}$, based on a moving window), estimated recalibration factor with exceedances, and transformed profits and losses Z_{t+1}.

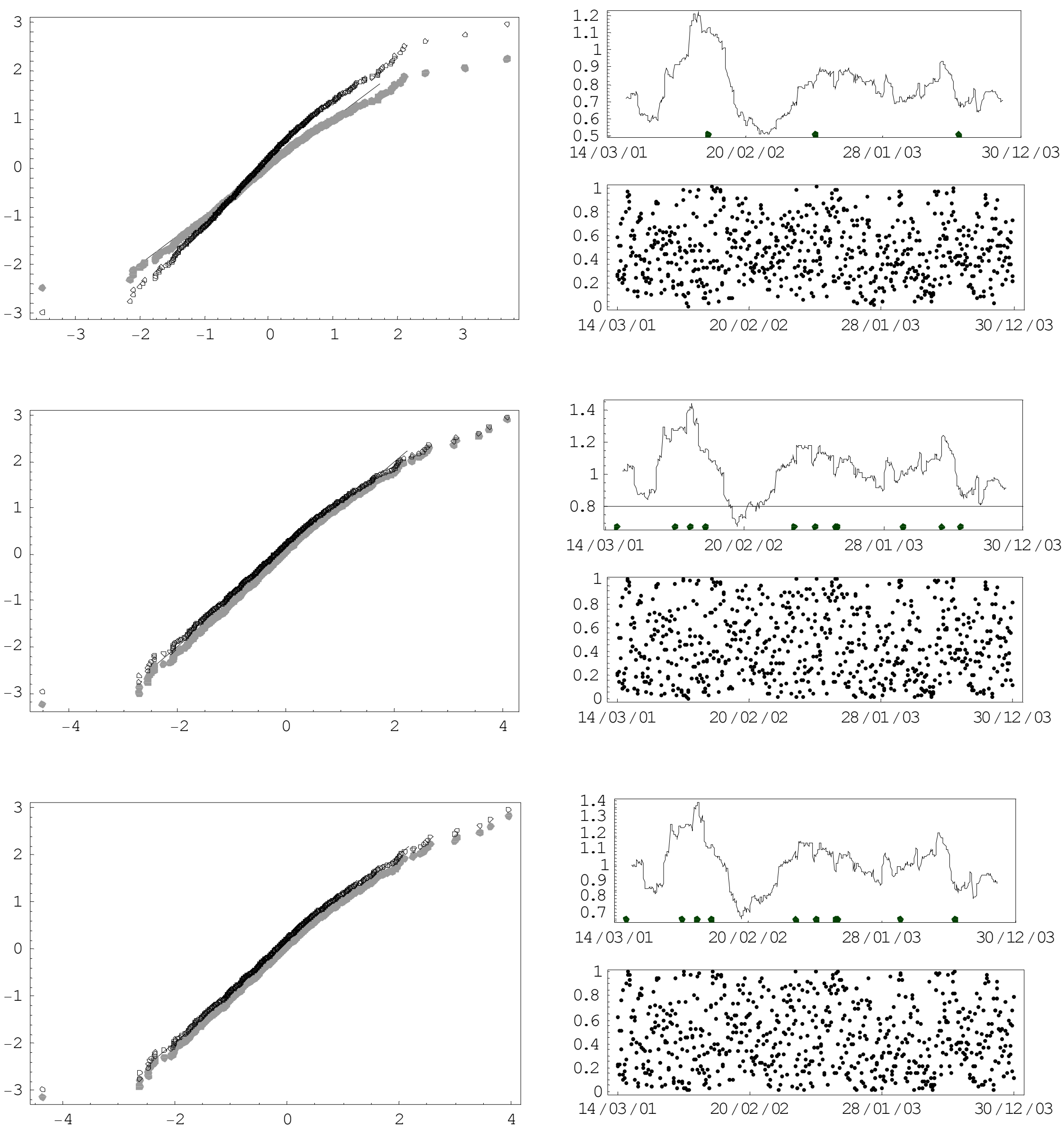

Figure 5: Validation of the aggregation approaches A4, A4a and A4b ("ac", from top to down): QQ-plot of standardized returns S_{t+1} (with (lighter) / without (dark) recalibration $\sigma_{rec,i}$, based on a moving window), estimated recalibration factor with exceedances, and transformed profits and losses Z_{t+1}.

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