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Forecast-error-based estimation of forecast uncertainty when the horizon is increased

Malte Knüppel

Deutsche Bundesbank, Wilhelm-Epstein-Straße 14, 60431 Frankfurt am Main,
Postfach 10 06 02, 60006 Frankfurt am Main

Tel +49 69 9566-0

Please address all orders in writing to: Deutsche Bundesbank,
Press and Public Relations Division, at the above address or via fax +49 69 9566-3077

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Non-technical summary

Research Question

How large is the precision that we can expect from a certain forecast, i.e. how large is the uncertainty surrounding the forecast? An easy way to assess this uncertainty is given by looking at the magnitude of the errors of past forecasts. However, in practice, sometimes there are hardly any past forecast errors for the horizon of interest available, because this horizon has been introduced quite recently. How can a reliable assessment of the forecast uncertainty be achieved in such a case?

Contribution

This paper suggests to additionally use the past forecast errors of other horizons, for which a larger sample is available, for the assessment of the uncertainty of the forecasts for the new horizon. The estimation method proposed relies on the fact that forecast errors of different horizons tend to move closely together. This method can be easily implemented, because it does not require the estimation of any additional parameters.

Results

In simulations, it turns out that the suggested estimation method works better than the usual approach where only the few past errors of the forecast horizon of interest are employed in order to assess this horizon's forecast uncertainty. The estimation method is also applied to the forecast errors of the Bank of England and of the Federal Open Market Committee. In both cases, the estimated forecast uncertainty appears more plausible than the forecast uncertainty obtained with the usual approach.

Nicht-technische Zusammenfassung

Fragestellung

Wie hoch ist die Präzision, die wir von einer Prognose erwarten dürfen, bzw. wie hoch ist die Unsicherheit, die diese Prognose umgibt? Eine einfache Art, um zu einer Einschätzung dieser Unsicherheit zu gelangen, besteht in der Betrachtung des Ausmaßes vergangener Prognosefehler. Es gibt jedoch Situationen in der Praxis, in denen nur sehr wenige vergangene Prognosefehler für den interessierenden Prognosehorizont verfügbar sind, weil dieser Prognosehorizont erst vor kurzem eingeführt wurde. Wie kann eine verlässliche Einschätzung der Prognoseunsicherheit in solch einem Fall erreicht werden?

Beitrag

In dieser Arbeit wird vorgeschlagen, auch die vergangenen Prognosefehler anderer Prognosehorizonte, für die eine größere Stichprobe vorliegt, zu verwenden, um zu einer Einschätzung der Unsicherheit für den neuen Prognosehorizont zu gelangen. Die vorgeschlagene Schätzmethode basiert darauf, dass die Prognosefehler verschiedener Horizonte meistens einen starken Gleichlauf aufweisen. Die Methode ist leicht implementierbar, da keine zusätzlichen Parameter geschätzt werden müssen.

Ergebnisse

In Simulationen erweist sich, dass die vorgeschlagene Schätzmethode bessere Ergebnisse liefert als der übliche Ansatz, bei dem nur die wenigen verfügbaren Prognosefehler des neuen Prognosehorizonts verwendet werden, um die Unsicherheit für diesen Horizont zu bestimmen. Auch bei einer Anwendung der Schätzmethode auf die Prognosefehler der Bank of England und des Federal Open Market Committee zeigt sich, dass die Schätzung der Prognoseunsicherheit plausiblere Ergebnisse liefert als der übliche Ansatz.

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Forecast-Error-Based Estimation of Forecast Uncertainty When the Horizon Is Increased*

Malte Knüppel

Deutsche Bundesbank

Abstract

Recently, several institutions have increased their forecast horizons, and many institutions rely on their past forecast errors to estimate measures of forecast uncertainty. This work addresses the question how the latter estimation can be accomplished if there are only very few errors available for the new forecast horizons. It extends upon the results of Knüppel (2014) in order to relax the condition on the data structure required for the SUR estimator to be independent from unknown quantities. It turns out that the SUR estimator of forecast uncertainty tends to deliver large efficiency gains compared to the OLS estimator (i.e. the sample mean of the squared forecast errors) in the case of increased forecast horizons. The SUR estimator is applied to the forecast errors of the Bank of England and the FOMC.

Keywords: multi-step-ahead forecasts, forecast error variance, SUR
JEL classification: C13, C32, C53

*Contact address: Malte Knüppel, Deutsche Bundesbank, Wilhelm-Epstein-Strasse 14, D-60431 Frankfurt am Main, Germany. Phone: ++49 69 9566 2324. E-Mail: malte.knueppel@bundesbank.de. Discussion Papers represent the authors' personal opinions and do not necessarily reflect the views of the Deutsche Bundesbank or its staff.

1 Introduction

In recent times, the forecast horizons of many important macroeconomic forecasts have increased. For example, since 2009, in addition to their forecasts for the current year and the next two years, the members of the Federal Open Market Committee (FOMC) have provided forecasts for the "longer run" in their Economic Projections, which might correspond to a horizon of about 5 years.[1] In the same year, in the Survey of Professional Forecasters conducted by the Federal Reserve Bank of Philadelphia, the forecast horizon for the annual forecasts of real GDP, the unemployment rate, and 3-month and 10-year Treasuries was extended from 2 to 4 years. In the Survey of Professional Forecasters conducted by the European Central Bank (ECB), since 2013, respondents have been asked about their 2-year ahead forecasts for real GDP growth, HICP inflation and the unemployment rate also in the first 2 quarters of the current year. Before 2013, forecasts for this horizon had been asked for only in the last 2 quarters.[2] Starting in 2014, the ECB staff and the Eurosystem staff have published 3-year-ahead forecasts for real GDP growth and HICP inflation in every quarter. Before 2014, such forecasts had only been made in the last quarter of the current year, while the largest forecast horizon in the first 3 quarters was equal to 2 years. The Bank of England (henceforth BoE) extended its forecast horizon for real GDP growth and CPI inflation from 8 to 12 quarters in 2004.

Many central banks and other institutions issuing forecasts provide measures of forecast uncertainty that are based on past forecast errors. The model-based approach for the estimation of forecast uncertainty, as described, for instance, in Ericsson (2002), is used only rarely because, as noted by Wallis (1989, pp. 55-56), "This approach is of little help to the practitioner. It neglects the contribution of the forecaster's subjective adjustments [...]. More fundamentally, the model's specification is uncertain. At any point in time competing models coexist, over time model specifications evolve, and there is no way of assessing this uncertainty. *Thus, the only practical indication of the likely margin of*

[1] The FOMC states "The longer-run projections are the rates of growth, inflation, and unemployment to which a policymaker expects the economy to converge over time–maybe in five or six years–in the absence of further shocks and under appropriate monetary policy."
http://www.federalreserve.gov/monetarypolicy/fomc_projectionsfaqs.htm

[2] In addition, the ECP SPF contains a longer-run type forecast for a forecast horizon of 4 or 5 years. When the survey started in 1999, this forecast was asked for only in the first quarter. Since 2001, it is included in every quarter.

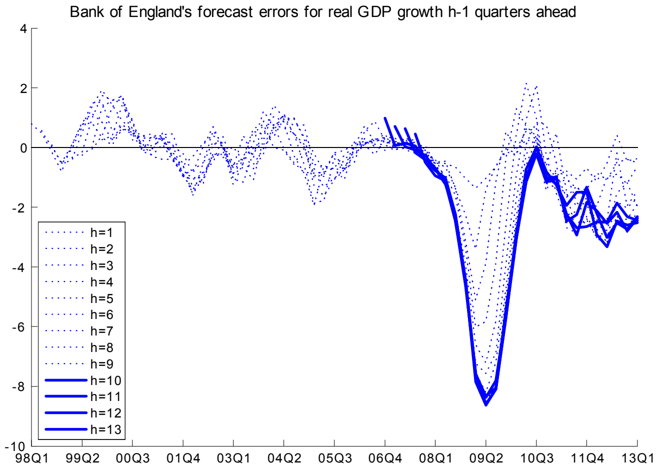

Figure 1: Forecast errors of the Bank of England for quarterly real GDP growth in the UK for the quarterly horizons $h = 1$ to $h = 13$. $h = 1$ corresponds to the nowcast.

future error is provided by the past forecast errors" [emphasis added]. Commonly, only the institutions' own forecast errors are used for estimating its forecast uncertainty. In general, the error-based forecast uncertainty measure is regarded as information about the *unconditional* uncertainty that can serve as a benchmark when statements about the current forecast uncertainty are made. For example, the FOMC members state whether the uncertainty attached to their current forecasts is larger than, smaller than, or broadly similar to the past forecast uncertainty.

If the past forecast uncertainty is to be estimated, the question arises how this can be accomplished in a reasonably precise manner if only very few observations are available, because a new forecast horizon was introduced only a short time ago. For example, for the FOMC forecasts, we will observe the first "longer run" forecast errors once the data for 2014 is released, if one assumes that the "longer run" corresponds to a forecast horizon of about 5 years. The first forecast errors of the ECB staff and the Eurosystem staff March forecasts will become available in 2017. Also for the BoE, whose growth forecast errors are displayed in Figure 1, the sample of forecast errors for the new forecast horizons still appears relatively short, taking their high persistence into account. This work is concerned with the estimation of forecast uncertainty for such horizons as soon as the first forecast errors are observed.

In the literature, the construction of uncertainty measures from past forecast errors goes back to the work of Williams and Goodman (1971). Recently, new contributions have been made inter alia by Lee and Scholtes (2014), Jordà, Knüppel and Marcellino (2013) and Knüppel (2014). Studies that focus on the uncertainty of central bank forecasts are provided by, for example, Reifschneider and Tulip (2007) and Tulip and Wallace (2012).[3]

Knüppel (2014) showed how to exploit the information contained in forecast errors of smaller forecast horizons for the estimation of the forecast uncertainty of larger horizons using the SUR estimator derived under the assumption of forecast optimality. This estimator relies on the correlations between forecast errors from different horizons for the same period, which are present for optimal as well as for non-optimal forecasts, and which also are a typical feature of empirical forecast errors like those displayed in Figure 1. However, a certain data structure labeled as recent-forecast-errors structure is required for this SUR estimator, leading to its independence from unknown parameters. This data structure differs from the structure that is observed when additional forecast horizons are introduced after the first forecasts were produced.

In the present study, these restrictions on the data structure are relaxed. It turns out that the SUR estimator continues to be independent from unknown parameters in the case of optimal forecasts, in circumstances not considered in that work. These circumstances include the case where new, larger forecast horizons are introduced. In this case, the SUR estimator delivers even larger efficiency gains than in the case of the recent-forecast-errors structure for optimal as well as for non-optimal forecasts.

2 The Estimation of Forecast Uncertainty

2.1 Fixed-Horizon Forecasts

A large part of the following expositions is taken from Knüppel (2014). Consider a stationary data-generating process with Wold representation given by

$$y_t = \mu + \sum_{i=0}^{\infty} b_i \varepsilon_{t-i}$$

[3] Reifschneider and Tulip (2007) use a different approach than the other studies mentioned, because they estimate their uncertainty measure from the errors of a *panel* of forecasters.

with $E[\varepsilon_t] = 0$, $E\left[\varepsilon_t^2\right] = \sigma^2$ and $b_0 = 1$. $E[\bullet]$ is the expectation operator. It is assumed that the fourth moment of ε_t exists, so that the kurtosis

$$\alpha = E\left[\varepsilon_t^4\right]/\sigma^4$$

is finite.

The optimal h-step-ahead forecast is

$$y_{t+h,t} = \mu + \sum_{i=0}^{\infty} b_{h+i}\varepsilon_{t-i}.$$

Hence, the h-step-ahead forecast error equals

$$e_{t+h,t} := y_t - y_{t+h,t} = \sum_{i=0}^{h-1} b_i \varepsilon_{t+h-i}. \quad (1)$$

Thus, $e_{t+h,t}$ is the error of the forecast made in period t for period $t+h$, and has a moving-average representation of order $h-1$.

The variance of the h-step-ahead forecast error is given by

$$E\left(e_{t+h,t}^2\right) = \sigma_h^2 = \sigma^2 \sum_{i=0}^{h-1} b_i^2.$$

The variances for all H^* forecast horizons of interest are collected in the $(H^* \times 1)$ vector σ^2, so that

$$\sigma^2 = \left(\sigma_1^2, \sigma_2^2, \ldots, \sigma_H^2\right)',$$

where H denotes the largest forecast horizon and $H^* \leq H$. The corresponding estimates of forecast uncertainty will be denoted as

$$\hat{\sigma}_m^2 = \left(\hat{\sigma}_{m,1}^2, \hat{\sigma}_{m,2}^2, \ldots, \hat{\sigma}_{m,H}^2\right)'$$

where m will refer to the estimation method used.

Since the DGP is stationary, the covariance between the squared forecast errors $e_{t_1+h_1,t_1}^2$ and $e_{t_1+h_2,t_2}^2$ tends to become very small when $|h_2 - h_1|$ becomes large. Assuming that the covariance actually equals zero if $h_1 \neq h_2$ gives rise to the possibility of estimating forecast uncertainty by seemingly unrelated regressions (SUR).

Under this assumption, and defining $p = t_2 - t_1$ and $q = t_4 - t_3$, the covariance of the squared forecast errors equals

$$\begin{aligned}\omega_s &= E\left[\left(e^2_{t_2,t_1} - E\left(e^2_{t_2,t_1}\right)\right)\left(e^2_{t_4,t_3} - E\left(e^2_{t_4,t_3}\right)\right)\right] \\ &= \begin{cases} 0 & \text{if } t_2 \neq t_4 \\ \sigma^4 (\alpha - 1) \sum\limits_{i=0}^{s-1} b_i^4 + 2\sigma^4 \sum\limits_{i=0}^{s-1} \sum\limits_{j=0, j\neq i}^{s-1} b_i^2 b_j^2 & \text{if } t_2 = t_4 \end{cases}\end{aligned} \tag{2}$$

with $s = \min(p, q)$ and using the convention $\sum_{i=0}^{0} \sum_{j=0,j\neq i}^{0} x_{ij} = 0$. If the kurtosis equals $\alpha = 3$, the last term in (2) simplifies to $2\sigma^4 \left(\sum_{i=0}^{s-1} b_i^2\right)^2$.

If the forecasts are non-optimal, the errors of forecasts for a certain period nevertheless tend to be strongly correlated due to the shock in that period. This makes the SUR estimation of forecast uncertainty a promising approach also in the case of non-optimal forecasts.

Concerning the data structure, suppose that the first forecast was made in period $t = 0$, and that the last available forecast errors are observed in period $t = T$. Define a $(T \times H)$ index matrix $\mathbf{J}$, where the element in the tth row and the jth column of $\mathbf{J}$, j_{th} is determined by the existence of the h-step-ahead forecast error for period t, i.e. by the error of the h-step-ahead forecast that was produced in period $t - h$. The element j_{th} equals

$$j_{th} = \begin{cases} 1 & \text{if } e_{t,t-h} \text{ exists} \\ 0 & \text{else} \end{cases}.$$

I assume that there is at least 1 1-step-ahead and 1 H-step-ahead forecast error. The corresponding $(T \times H)$ matrix of squared forecast errors is given by $\mathbf{E}^2$

$$\mathbf{E}^2 = \begin{bmatrix} \tilde{e}^2_{1,0} & \tilde{e}^2_{1,-1} & \cdots & \tilde{e}^2_{1,1-H} \\ \tilde{e}^2_{2,1} & \tilde{e}^2_{2,0} & \cdots & \tilde{e}^2_{2,2-H} \\ \tilde{e}^2_{3,2} & \tilde{e}^2_{3,1} & \cdots & \tilde{e}^2_{3,3-H} \\ \vdots & \vdots & \ddots & \vdots \\ \tilde{e}^2_{T,T-1} & \tilde{e}^2_{T,T-2} & \cdots & \tilde{e}^2_{T,T-H} \end{bmatrix}$$

with elements

$$\tilde{e}^2_{t,t-h} = \begin{cases} e^2_{t,t-h} & \text{if } e^2_{t,t-h} \text{ exists} \\ c & \text{else} \end{cases},$$

where c is an arbitrary value that will receive a weight of 0 in the estimation. If only the elements $\tilde{e}^2_{t,t-h}$ with $t-h<0$ equal c, the data structure is equivalent to the "recent forecast errors" structure considered in Knüppel (2014).

Hence, for example, if 1- and 2-step-ahead forecasts were produced since $t=0$, if the last observation came from $t=7$, if additional 4-step-ahead forecasts were produced in $t=1$ and $t=3$, and if no forecast errors were available for $t=3$, $\mathbf{J}$ and $\mathbf{E}^2$ would equal

$$\mathbf{J} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix}, \quad \mathbf{E}^2 = \begin{bmatrix} e^2_{1,0} & c & c & c \\ e^2_{2,1} & e^2_{2,0} & c & c \\ c & c & c & c \\ e^2_{4,3} & e^2_{4,2} & c & c \\ e^2_{5,4} & e^2_{5,3} & c & e^2_{5,1} \\ e^2_{6,5} & e^2_{6,4} & c & c \\ e^2_{7,6} & e^2_{7,5} & c & e^2_{7,3} \end{bmatrix}. \tag{3}$$

In order to estimate a model with correlated error terms by SUR, one needs to construct a covariance matrix and a regressor matrix of the error terms collected in $\mathbf{E}^2$. If $\mathbf{J}$ only consists of 1's, the $(TH \times TH)$ SUR covariance matrix of $vec\left(\mathbf{E}^2\right)$ is given by

$$\mathbf{\Omega}_{SUR} = \begin{bmatrix} \omega_1 & \omega_1 & \omega_1 & \dots & \omega_1 \\ \omega_1 & \omega_2 & \omega_2 & \dots & \omega_2 \\ \omega_1 & \omega_2 & \omega_3 & \dots & \omega_3 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \omega_1 & \omega_2 & \omega_3 & \dots & \omega_H \end{bmatrix} \otimes \mathbf{I}_T. \tag{4}$$

where $\mathbf{I}_n$ denotes the $(n \times n)$ identity matrix. Accounting for the possibility that elements of $\mathbf{J}$ can equal 0 requires modifications of the SUR covariance matrix. Denote the number of columns of $\mathbf{J}$ with at least one element equal to 1 by H^*, and let the unity vector $\mathbf{e}_{i,H}$

denote the ith column of $\mathbf{I}_H$. Then define the $(H \times H^*)$ selection matrix $\mathbf{S}$ as

$$\mathbf{S} = \begin{bmatrix} \mathbf{e}_{1,H} & \mathbf{e}_{2,H} & \dots & \mathbf{e}_{H,H} \end{bmatrix}$$

where the vector $\mathbf{e}_{i,H}$ is contained only if the ith column of $\mathbf{J}$ has at least one element equal to 1.

Moreover, define the $(TH \times TH)$ diagonal matrix

$$\mathbf{D} = diag\left(vec\left(\mathbf{J}\right)\right).$$

Then the $(TH^* \times TH^*)$ SUR covariance matrix to be used for the estimation is given by

$$\boldsymbol{\Omega}_{SUR}^* = \left(\mathbf{S}' \otimes \mathbf{I}_T\right)\left(\mathbf{D}\boldsymbol{\Omega}_{SUR}\mathbf{D}' + \mathbf{I}_{TH} - \mathbf{D}\right)\left(\mathbf{S}' \otimes \mathbf{I}_T\right)'$$

where $\otimes$ denotes the Kronecker product. The modification involving the $\mathbf{D}$ matrix is required in order to account for the zero elements in $\mathbf{J}$. The modification using the selection matrix $\mathbf{S}$ deletes all elements related to forecast horizons for which no forecast errors exist.

Now define the $(T \times H^*)$ matrices

$$\mathbf{J}^* = \mathbf{J}\mathbf{S}$$
$$\mathbf{E}^{*2} = \mathbf{E}^2\mathbf{S}$$

and the $(TH^* \times H^*)$ regressor matrix

$$\mathbf{X} = \left(\mathbf{I}_{H^*} \otimes \mathbf{1}_T\right) \odot \left(\mathbf{1}_{H^*} \otimes \mathbf{J}^*\right)$$

where $\mathbf{1}_n$ denotes an $(n \times 1)$ vector of ones, and $\odot$ denotes the Hadamard product.

Then the SUR estimator of forecast uncertainty is calculated as

$$\hat{\sigma}_{SUR}^2 = \left(\mathbf{X}'\boldsymbol{\Omega}_{SUR}^{*-1}\mathbf{X}\right)^{-1}\mathbf{X}'\boldsymbol{\Omega}_{SUR}^{*-1}vec\left(\mathbf{E}^{*2}\right) \qquad (5)$$

and the OLS estimator, which yields the sample means, as

$$\hat{\sigma}^2_{OLS} = \left(\mathbf{X}'\mathbf{X}\right)^{-1}\mathbf{X}'vec\left(\mathbf{E}^{*2}\right). \tag{6}$$

For the following considerations, it is helpful to define the matrix

$$\mathbf{A} = \left(\mathbf{X}'\mathbf{\Omega}^{*-1}_{SUR}\mathbf{X}\right)^{-1}\mathbf{X}'\mathbf{\Omega}^{*-1}_{SUR} \tag{7}$$

so that $\mathbf{XA}$ is the projection matrix.

Conjecture 1 *If for all t and for all $h > 1$, $j^*_{th} = 1$ implies that $j^*_{ti} = 1$ for every integer i with $0 < i < h$, then $\mathbf{A}$ only depends on $\mathbf{J}$.*

This conjecture states that, if the forecast error $e_{t,t-h}$ is available, then the availability of all other forecast errors for the same period from smaller horizons $e_{t,t-h+1}, e_{t,t-h+2}, \ldots$ implies that $\mathbf{A}$ only depends on known quantities. Neither the dynamics of y_t nor the kurtosis of the shocks affect the estimator. This result also holds if $e_{t,t-h+i}$ (with $i > 0$) is not available, but the $(h-i)$th column of $\mathbf{J}$ contains 0's only, i.e. if there are no forecast errors for the horizon $h - i$. A proof of the conjecture for the special case of $H^* = 2$ can be found in Appendix A.1.

The condition of this conjecture is fulfilled in the case where a new larger forecast horizon is introduced after the first forecasts were produced. A simple example is given by the presence of T 1-step-ahead forecast errors $e_{t,t-1}$ with $t = 1, 2, ..., T$ and 1 2-step-ahead forecast error $e_{T,T-2}$, which would lead to the matrix

$$\mathbf{J} = \mathbf{J}^* = \begin{bmatrix} \mathbf{1}_{T-1} & \mathbf{0}_{T-1} \\ 1 & 1 \end{bmatrix}. \tag{8}$$

The condition of the conjecture is fulfilled because, in the last row, $j^*_{T2} = 1$ and $j^*_{T1} = 1$. The condition would be violated if $j^*_{T1} = 0$. In the example given by (3), the condition is fulfilled as well.

If the condition is fulfilled, the matrix $\mathbf{A}$ can be determined without using (5).[4] In

[4] Like the conjecture itself, the validity of the construction proposed has not been confirmed analytically for $H^* > 2$. However, in simulations considering many different matrices $\mathbf{J}$ where the condition of the

order to do so, partition $\mathbf{A}$ into the H^* submatrices $\mathbf{A}_{h^*}$ of dimension $(H^* \times T)$ with $h^* = 1, 2, ..., H^*$.

$$\mathbf{A} = \begin{bmatrix} \mathbf{A}_1 & \mathbf{A}_2 & \ldots & \mathbf{A}_{H^*} \end{bmatrix}$$

Then, for each h^*, calculate

$$w_{h^*} = \left(\mathbf{1}_T' \mathbf{J}^* \mathbf{e}_{h^*,H^*}\right)^{-1}.$$

If $h^* < H^*$, determine

$$\begin{aligned} \mathbf{O}_{h^*} &= (\mathbf{J}^* \mathbf{e}_{h^*,H^*}) \odot (\mathbf{J}^* \mathbf{e}_{h^*+1,H^*}) \\ \mathbf{N}_{h^*} &= (\mathbf{J}^* \mathbf{e}_{h^*,H^*}) \odot (\mathbf{1}_T - \mathbf{J}^* \mathbf{e}_{h^*+1,H^*}) \\ w_{h^*}^o &= -w_{h^*} \left(\mathbf{1}_T' \mathbf{N}_{h^*}\right) \left(\mathbf{1}_T' \mathbf{O}_{h^*}\right)^{-1}. \end{aligned}$$

$\mathbf{O}_{h^*}$ is a vector that equals 1 in those periods t where a forecast error exists in the (h^*)th and in the $(h^* + 1)$th column of $\mathbf{E}^{*2}$ (periods with 'overlapping' forecast errors) and 0 otherwise. $\mathbf{N}_{h^*}$ is a vector that equals 1 in those periods t where a forecast error exists in the (h^*)th column, but not in the $(h^* + 1)$th column of $\mathbf{E}^{*2}$ (periods with 'non-overlapping' forecast errors) and 0 otherwise. w_{h^*} and $w_{h^*}^o$ are scalar weights. Use these quantities to calculate $\mathbf{A}_{h^*}$ as

$$\mathbf{A}_{h^*} = \begin{bmatrix} \mathbf{0}_{(h^*-1)\times T} \\ w_{h^*} \left(\mathbf{J}^* \mathbf{e}_{h^*,H^*}\right)' \\ w_{h^*}^o \mathbf{O}_{h^*}' + w_{h^*} \mathbf{N}_{h^*}' \\ \vdots \\ w_{h^*}^o \mathbf{O}_{h^*}' + w_{h^*} \mathbf{N}_{h^*}' \end{bmatrix}. \tag{9}$$

If $h^* = H^*$, $\mathbf{A}_{H^*}$ equals

$$\mathbf{A}_{H^*} = \begin{bmatrix} \mathbf{0}_{(H^*-1)\times T} \\ w_{H^*} \left(\mathbf{J}^* \mathbf{e}_{H^*,H^*}\right)' \end{bmatrix}. \tag{10}$$

Note that w_{h^*} is equal to the OLS weight for an observation belonging to the (h^*)th column of $\mathbf{E}^{*2}$. For the example (8), one hence obtains $w_1 = 1/T$ and $w_2 = 1$. For the

conjecture is fulfilled, and considering many different values of α and the b_i's, the construction described always yielded the same result as (7).

same example, $\mathbf{A}$ is given by

$$\mathbf{A} = \begin{bmatrix} (1/T)\cdot\mathbf{1}'_{T-1} & 1/T & \mathbf{0}'_{T-1} & 0 \\ (1/T)\cdot\mathbf{1}'_{T-1} & -(T-1)/T & \mathbf{0}'_{T-1} & 1 \end{bmatrix}.$$

This example illustrates how the SUR estimator uses information from smaller forecast horizons to calculate uncertainty for larger horizons. The 1-step-ahead error that is affected by the same shock as the 2-step-ahead error receives a negative weight in the estimation of the 2-step-ahead uncertainty. In order for the estimator to remain consistent, the $T-1$ other 1-step-ahead errors must receive weights which equal $-1/(T-1)$ times the negative weight. Note that for large T, $\hat{\sigma}^2_{SUR,2} \approx \hat{\sigma}^2_{OLS,1} - e^2_{T,T-1} + e^2_{T,T-2}$.

This example also shows that the SUR estimator can, in principle, generate negative estimates for $\hat{\sigma}^2_{SUR,h^*}$. In order to avoid such results, one could consider the modified estimator

$$\hat{\sigma}^2_{SUR} = \max\left(\mathbf{0}_{H^*}, \mathbf{A}vec\left(\mathbf{E}^{*2}\right)\right)$$

where $\max(\cdot)$ is applied elementwise. This estimator can be expected to be upward biased, but to yield a lower mean squared error than $\hat{\sigma}^2_{SUR}$.[5]

2.2 Fixed-Event Forecasts

There are at least two important types of fixed-event forecasts. For example, the ECB staff and the Eurosystem staff forecasts are made for changes of annual averages several times each year. In contrast to that, the FOMC forecasts are concerned with changes from the 4th quarter of a certain year to the 4th quarter of the following year (henceforth q4/q4 forecasts).[6] In what follows, I focus on the latter type of fixed-event forecasts, assuming that 4 forecasts per year are made.[7] Adapting the formulas to the situation where there are monthly forecasts of changes from the 12th month of a certain year to the 12th month of the following year is straightforward.

[5] Alternatively, one could also use the function $\max\left(\lambda\hat{\sigma}^2_{OLS}, \mathbf{A}vec\left(\mathbf{E}^{*2}\right)\right)$ with $0 \leq \lambda \leq 1$.

[6] In both cases, the forecast for the unemployment rate concerns the level itself, and not the change in the level.

[7] If the fixed-event forecasts are concerned with annual averages, the SUR estimator is found to depend on unknown parameters. It remains to be investigated if efficiency gains can still be obtained simply by assuming certain values for the b_i's and for α, for example by setting $\alpha = 3$ and $b_i = c$ for $i > 0$ with c close to zero.

The variable of interest x_s is defined by

$$x_s = y_t + y_{t-1} + y_{t-2} + y_{t-3}$$

where s refers to the year and t denotes the last quarter in year s. x_s denotes the change from the 4th quarter in $s-1$ to the 4th quarter of s. y_t denotes the change from quarter $t-1$ to quarter t. The corresponding forecast errors for x_s are given by

$$\begin{aligned}
u_{s,t-1} &= e_{t,t-1} \\
u_{s,t-2} &= e_{t,t-2} + e_{t-1,t-2} \\
u_{s,t-3} &= e_{t,t-3} + e_{t-1,t-3} + e_{t-2,t-3} \\
u_{s,t-4} &= e_{t,t-4} + e_{t-1,t-4} + e_{t-2,t-4} + e_{t-3,t-4} \\
u_{s,t-5} &= e_{t,t-5} + e_{t-1,t-5} + e_{t-2,t-5} + e_{t-3,t-5} \\
&\vdots \\
u_{s,t-H} &= e_{t,t-H} + e_{t-1,t-H} + e_{t-2,t-H} + e_{t-3,t-H}
\end{aligned}$$

where H is typically a multiple of 4. For example, if forecasts for x_s are made for the current and the next year in the 1st, 2nd, and 3rd quarter, and for the next year and the year after that in the 4th quarter, if the first forecast was made in the 4th quarter of the year $s = 0$, the last realization is observed for the year $s = 2$, and the last quarter of the year $s = 2$ is denoted as $t = 8$, then one obtains

$$\mathbf{J} = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

$$\mathbf{E}^2 = \begin{bmatrix} u_{1,3}^2 & u_{1,2}^2 & u_{1,1}^2 & u_{1,0}^2 & c & c & c & c \\ u_{2,7}^2 & u_{2,6}^2 & u_{2,5}^2 & u_{2,4}^2 & u_{2,3}^2 & u_{2,2}^2 & u_{2,1}^2 & u_{2,0}^2 \end{bmatrix}.$$

In this example, $H = 8$.

If $\mathbf{J}$ only contains 1's, the SUR covariance matrix of $vec\left(\mathbf{E}^2\right)$ equals

$$\mathbf{\Omega}_{SUR} = \begin{bmatrix} \omega_1 & \omega_1 & \omega_1 & \omega_1 & \omega_1 & \omega_1 & \cdots & \omega_1 \\ \omega_1 & \sum_{i=1}^{2}\omega_i & \sum_{i=1}^{2}\omega_i & \sum_{i=1}^{2}\omega_i & \sum_{i=1}^{2}\omega_i & \sum_{i=1}^{2}\omega_i & \cdots & \sum_{i=1}^{2}\omega_i \\ \omega_1 & \sum_{i=1}^{2}\omega_i & \sum_{i=1}^{3}\omega_i & \sum_{i=1}^{3}\omega_i & \sum_{i=1}^{3}\omega_i & \sum_{i=1}^{3}\omega_i & \cdots & \sum_{i=1}^{3}\omega_i \\ \omega_1 & \sum_{i=1}^{2}\omega_i & \sum_{i=1}^{3}\omega_i & \sum_{i=1}^{4}\omega_i & \sum_{i=1}^{4}\omega_i & \sum_{i=1}^{4}\omega_i & \cdots & \sum_{i=1}^{4}\omega_i \\ \omega_1 & \sum_{i=1}^{2}\omega_i & \sum_{i=1}^{3}\omega_i & \sum_{i=1}^{4}\omega_i & \sum_{i=2}^{5}\omega_i & \sum_{i=2}^{5}\omega_i & \cdots & \sum_{i=2}^{5}\omega_i \\ \omega_1 & \sum_{i=1}^{2}\omega_i & \sum_{i=1}^{3}\omega_i & \sum_{i=1}^{4}\omega_i & \sum_{i=2}^{5}\omega_i & \sum_{i=3}^{6}\omega_i & \cdots & \sum_{i=3}^{6}\omega_i \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \omega_1 & \sum_{i=1}^{2}\omega_i & \sum_{i=1}^{3}\omega_i & \sum_{i=1}^{4}\omega_i & \sum_{i=2}^{5}\omega_i & \sum_{i=3}^{6}\omega_i & \cdots & \sum_{i=H-3}^{H}\omega_i \end{bmatrix} \otimes \mathbf{I}_T. \tag{11}$$

While the SUR covariance matrix differs from its counterpart in the fixed-horizon case, all other formulas continue to apply. Most importantly, the conjecture made in the fixed-horizon case is valid in the fixed-event case as well, and the matrix $\mathbf{A}$ for the fixed-event case can be determined in the same manner as in the fixed-horizon case, i.e. by using (9) and (10).

The latter result points to a more general applicability of the conjecture. Indeed, simulations suggest that it always appears to hold if $\mathbf{\Omega}_{SUR}$ can be expressed as

$$\mathbf{\Omega}_{SUR} = \mathbf{\Psi} \otimes \mathbf{I}_T$$

with $\mathbf{\Psi}$ having full rank and the elements of $\mathbf{\Psi}$ having the property

$$\psi_{ij} = \psi_{\min(i,j)\,\min(i,j)}$$

with $\psi_{ij} \in \mathbb{R}$. Obviously, the $\mathbf{\Psi}$ matrices in (4) and (11) have this property.

3 Monte Carlo Simulations

As shown by Grenander and Rosenblatt (1957), OLS estimation is asymptotically as efficient as SUR estimation if the regressors consist exclusively of constants, so that SUR estimation can be helpful in small samples only. If a new forecast horizon is introduced, the first sample that can be used for estimating its forecast uncertainty only contains one observation for this horizon and, hence, can certainly be considered as small. In order to illustrate this, Monte Carlo simulations are employed in what follows. In these simulations, the forecaster produces forecasts for y_{t+h} using the zero-mean first-order autoregressive (henceforth AR(1)) model

$$\hat{y}_{t+h} = \hat{\rho} y_t$$

The data-generating processes (DGPs) for y_t considered are described by

$$y_t = \mu + x_t$$
$$x_t = \rho_1 x_{t-1} + \rho_2 x_{t-2} + \varepsilon_t$$

with ε_t iid normal and $Var(y_t) = 1$. The four parametrizations investigated and their descriptions are given by

optimal:	$\mu = 0,\ \rho_1 = \hat{\rho},\ \rho_2 = 0$
biased-mean:	$\mu = 2,\ \rho_1 = \hat{\rho},\ \rho_2 = 0$
biased-AR(1)-coefficient:	$\mu = 0,\ \rho_1 = \hat{\rho} + 0.2,\ \rho_2 = 0$
AR(1)-model for AR(2)-DGP:	$\mu = 0,\ \rho_1 = -\hat{\rho}(\rho_2 - 1),\rho_2 = -0.5$

Note that especially the biased-mean forecast is severely misspecified, having a mean that differs from the true value by 2 standard deviations of y_t.

In the simulations, forecasts for these processes are made for the sample sizes $T = 1, 2, ..., 40$. Starting in period $t = 0$, forecasts for 1 to 4 periods ahead are produced. Starting in period $t = 10$, additionally, forecasts for 5 periods ahead are made. σ^2 is estimated recursively as the sample size grows, using the SUR estimator (5) and the OLS estimator (6). The forecast uncertainty for $h = 5$ can be estimated only if $T \geq 15$. For $\hat{\rho}$, the values 0.5, 0.8 and 0.95 are considered in the cases of optimal forecasts, biased-mean forecasts, and the AR(1) forecasts for the AR(2)-DGP. In the case of the biased AR(1)-coefficient, the values 0.3, 0.6 and 0.75 are employed for $\hat{\rho}$.

The measure of efficiency gains of the SUR estimator for the 5-step-ahead forecasts is defined by

$$100 \ln \left(\sqrt{E \left[\left(\hat{\sigma}^2_{OLS,5} - \sigma^2_5 \right)^2 \right] \div E \left[\left(\hat{\sigma}^2_{SUR,5} - \sigma^2_5 \right)^2 \right]} \right).$$

Hence, values larger than 0 indicate efficiency gains of the SUR estimator. The unit is percent (with respect to the standard deviation of the OLS estimator).

As shown in Figure 2, large efficiency gains are possible with the SUR estimator. With $\rho = 0.5$ they can exceed 100 percent when the sample size is small. They decrease with the sample size, but remain elevated even when the full sample is available. With $\rho = 0.8$, the efficiency gains are smaller but still large for all sample sizes considered, and they also decrease with the sample size. With $\rho = 0.95$, the efficiency gains are still notable in almost all cases and can reach more than 40%. The efficiency gains first increase with the sample size, and then decrease as the sample size becomes larger. If the forecasts are not optimal, the efficiency gains tend to be smaller than in the case of optimal forecasts, but the differences are relatively small.

However, since the SUR estimator neglects the autocorrelation of forecast errors, the OLS estimator can be more efficient in certain situations. These cases are characterized by strongly autocorrelated forecast errors in connection with very few observations before the increase in the forecast horizon takes place, and very few subsequent observations.

This situation is illustrated in Figure 3, where the same settings as in the previous simulations are used, but the 5-step-ahead forecasts do not start in $t = 10$, but in $t = \max(0, T - 5)$. Thus, for every sample size with $T \geq 5$ considered, there is only a single 5-step-ahead forecast error available. If the sample size equals $T = 5$ or $T = 6$, that is if the 5-step-ahead forecasts start in $t = 0$ (the special case of the recent-forecast-errors structure) or $t = 1$, efficiency losses of the SUR estimator can be observed if the DGP is sufficiently persistent. However, in the cases which are more relevant in the context of this paper, i.e. the cases where the forecast horizon is increased several periods after the first forecasts for lower horizons are made, the efficiency gains are always positive and often large. Hence, employing the SUR estimator in the case of an increased forecast horizon is even more recommendable than in the case of the recent-forecast-errors structure analyzed in Knüppel (2014).

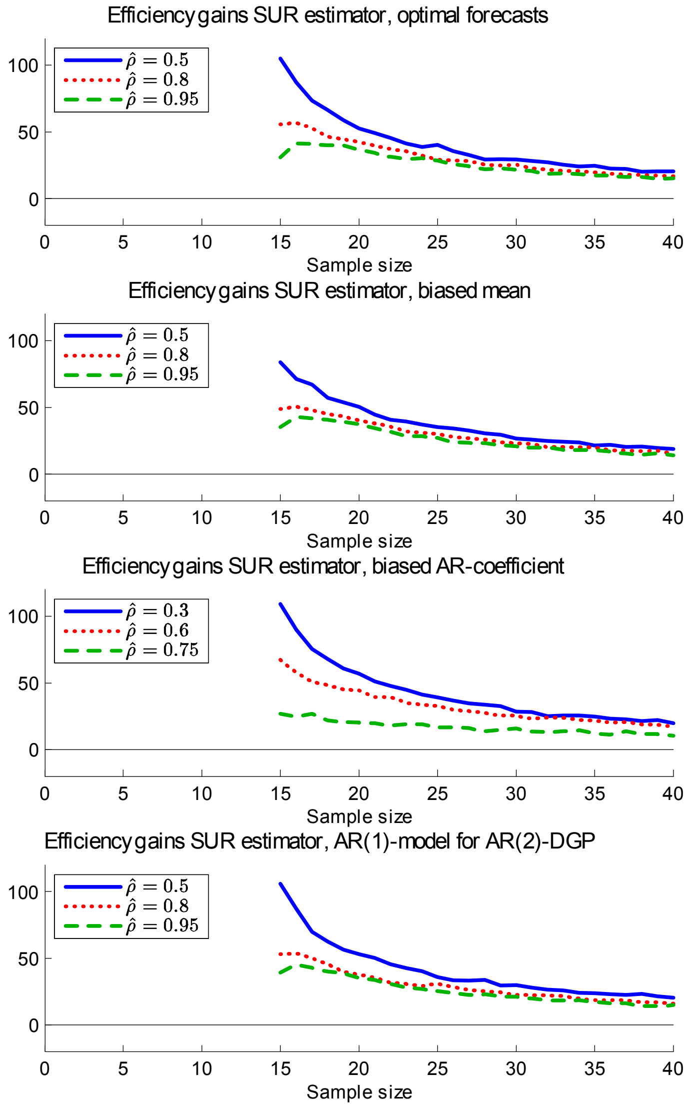

Figure 2: Efficiency gains of SUR estimator (5) for new forecast horizon $h = 5$, for which first forecast is made in $t = 10$. Sample size increases from $T = 1$ to $T = 40$. Results are based on 10,000 Monte Carlo simulations.

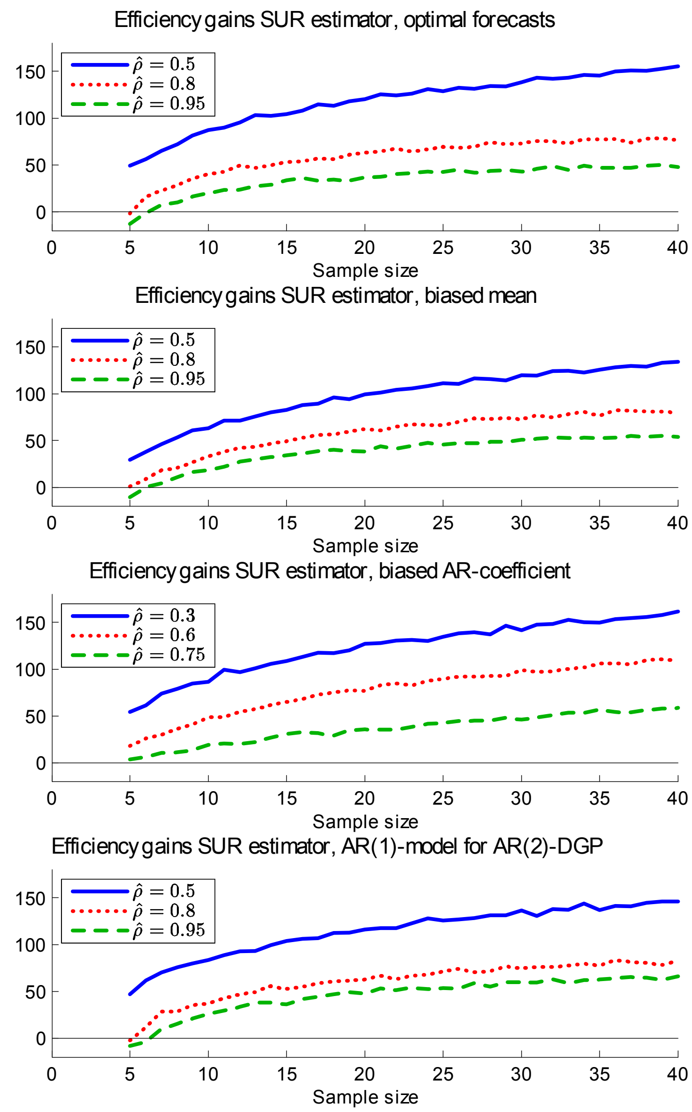

Figure 3: Efficiency gains of SUR estimator (5) for new forecast horizon $h = 5$, for which first forecast is made in $t = \max(0, T - 5)$, so that at most 1 forecast error for this horizon is available. Sample size increases from $T = 1$ to $T = 40$. Results are based on 10,000 Monte Carlo simulations.

4 Applications

In the following, two applications of the SUR estimator are presented. They are based on the forecasts for real GDP growth of the BoE for the UK, and of the FOMC forecasts for the US. A common feature of both forecasts is the presence of nowcasts, i.e. the smallest forecast horizon corresponds to a forecast for the current period. Thus, when forecasts are made up to $H-1$ periods ahead, forecasts for H horizons are available. The BoE forecasts have a fixed horizon, whereas the FOMC forecasts are fixed-event forecasts.

Concerning the data from the BoE, quarterly growth forecasts have been available for up to 8 quarters ahead since 1998q1. In 2004q3, the largest forecast horizon increased from 8 to 12 quarters. Thus, the first forecast errors for these horizons became available in 2006q4 to 2007q3, while up to 35 forecast errors for the smaller horizons already existed in 2006q3. The forecast errors are calculated using second vintages from the real-time database of the BoE which ends in 2013q1.

The full-sample estimates of the forecast uncertainty σ^2 are depicted in Figure 4. Clearly, the OLS results for $h \geq 10$ are affected by the fact that they are estimated based on a subset of the full sample where the (squared) forecast errors tended to be large. Since the SUR estimator employs the entire sample for the estimation for all horizons, the forecast uncertainty evolves relatively smoothly with the increase in the forecast horizon.

The recursive estimates of forecast uncertainty which start in 2006q4 are displayed in Figure 5. While the SUR and OLS estimates attained similar values from 2006 to the beginning of 2008, the large forecast errors for the end of 2008 and for 2009 lead to a strong increase in the OLS estimates. The increase is more subdued for the SUR estimates, because they employ a larger sample. The OLS estimates peak in 2009q4 and decrease strongly afterwards, while the SUR estimates remain relatively stable after 2009q4.

Concerning the data from the FOMC, I use the quarterly forecasts for the q4/q4 growth rate that have been available since October 2007.[8] These forecasts, the so-called Summaries of Economic Projections, contain nowcasts, 1-year-ahead, 2-year-ahead, and, depending on the quarter when the forecast is made, also 3-year-ahead forecasts. Moreover, since the 1st quarter of 2009, the FOMC has published "longer-run" forecasts, which are

[8] The forecasts were previously only made on a semi-annual basis.

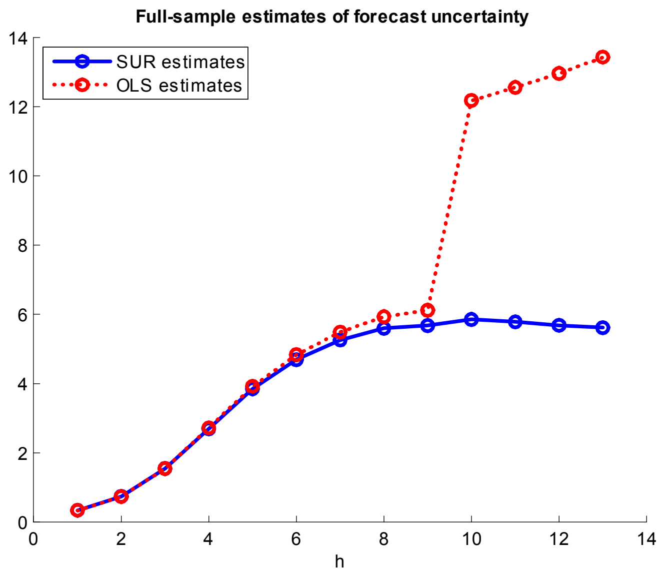

Figure 4: Estimates of the uncertainty σ^2 concerning the Bank of England's forecasts for quarterly real GDP growth in the UK for the quarterly horizons $h = 1$ to $h = 13$. $h = 1$ corresponds to the nowcast.

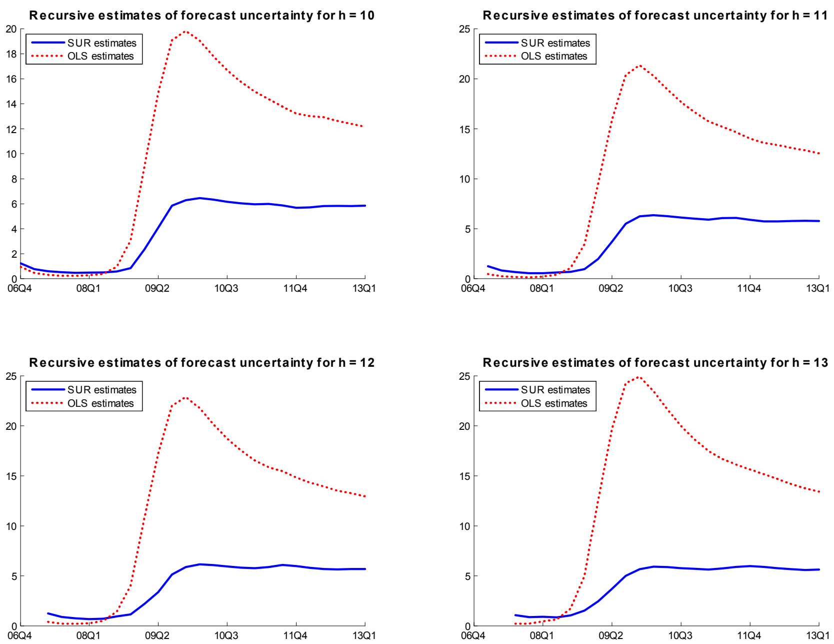

Figure 5: Evolution of the recursive estimates of the uncertainty σ^2 concerning the Bank of England's forecasts for real GDP growth in the UK for the quarterly horizons $h = 10, 11, 12, 13$

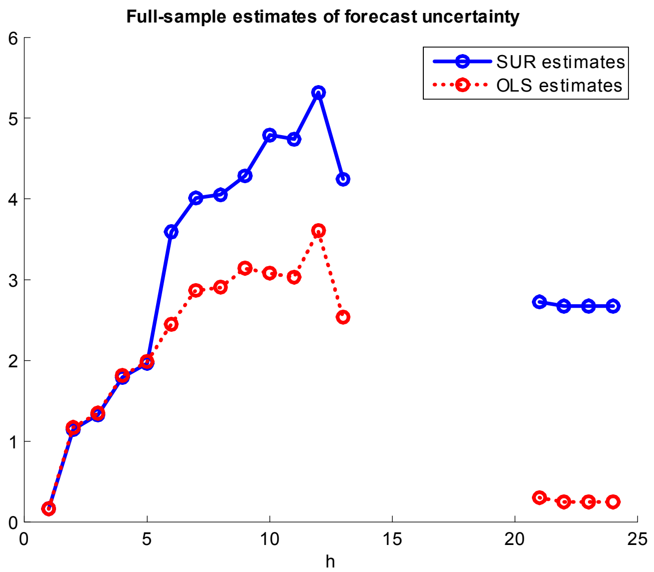

Figure 6: Estimates of the uncertainty σ^2 concerning the FOMC's fixed-event forecasts for q4/q4 real GDP growth in the US for the quarterly horizons $h = 1$ to $h = 24$. $h = 1$ corresponds to the nowcast, i.e. the forecast made in the 4th quarter for the current year. $h = 21, 22, 23, 24$ correspond to the long-run forecasts. It is assumed that the q4/q4 growth rate for 2014 equals 2.1%.

assumed to represent 5-year-ahead forecasts. Thus, this is a case where $H^* < H$.[9] Hence, the largest forecast horizon is 23 quarters, corresponding to the long-run forecast made in the first quarter of the year. The forecasts are given as ranges only. I calculate a point forecast as the mean of the lower and the upper end of the "central tendency" (henceforth referred to as mean forecast).

Once the data for 2014 is released, the first errors for the long-run forecasts can be calculated. In what follows, I assume that the q4/q4 growth rate for 2014 equals 2.1%, which is the mean forecast by the FOMC in September 2014. Moreover, I assume that the mean forecasts in December 2014 will be equal to the mean forecast in September 2014. Under these assumptions, the estimates of forecast uncertainty σ^2 depicted in Figure 6 are obtained.

Obviously, the OLS estimates of long-run forecast uncertainty deliver implausibly small

[9] The timing of the FOMC forecasts changed in 2013. From October 2007 until mid-2012, forecasts were made in January, April, June, and October/November. Since mid-2012, they have been produced in March, June, September and December. One could try to construct March and September forecasts before mid-2012 by interpolation in order to achieve a better synchronization of the forecast dates, but here I abstain from doing so. In what follows, the pre-2012 forecasts from June are considered as forecasts from the 3rd quarter. In 2012, 5 forecasts were actually made. The April forecasts are not part of the data set used here.

values, which is due to the fact that they are based on one observation only. The SUR long-run estimates are larger because they make use of all observations. Nevertheless, both estimators yield values that are smaller than many of the estimates for smaller horizons, which does not appear very plausible and indicates the need for a larger sample.

5 Conclusion and Outlook

In this paper, the restriction on the data structure that was imposed in Knüppel (2014) for the SUR estimation of forecast uncertainty for multi-step-ahead forecasts, labeled as recent-forecast-errors structure, is relaxed. It turns out that the SUR estimator continues to have the useful property found in Knüppel (2014), namely its independence from unknown parameters in the case of optimal forecasts, in circumstances not considered in that work. These circumstances include the empirically relevant case where new, larger forecast horizons are introduced. The SUR estimator is found to provide potentially large efficiency gains in this case which exceed those observed in the recent-forecast-errors case. Applications to the fixed-horizon growth forecasts of the Bank of England and the fixed-event growth forecasts of the FOMC illustrate the usefulness of the proposed SUR estimator.

A Appendix

A.1 Proof of Conjecture for $H^* = 2$

Suppose that $H^* = 2$. In this case, the proof of the conjecture for parameter independence is very similar to the one in Knüppel (2014) which follows the setup of Im (1994). It requires that all rows that only contain 0's be deleted from $\mathbf{X}$, and that the remaining rows are reorganized (if necessary) such that the resulting matrix $\tilde{\mathbf{X}}$ can be written as

$$\tilde{\mathbf{X}} = \begin{bmatrix} \mathbf{1}_T & \mathbf{0}_T \\ \mathbf{0}_{T-M} & \mathbf{1}_{T-M} \end{bmatrix} = \begin{bmatrix} \mathbf{1}_M & \mathbf{0}_M \\ \mathbf{1}_{T-M} & \mathbf{0}_{T-M} \\ \mathbf{0}_{T-M} & \mathbf{1}_{T-M} \end{bmatrix}.$$

where M is the number of periods where only 1-step errors are available, so that $T-M$ is the number of 2-step errors available. The corresponding SUR covariance matrix equals

$$\begin{aligned}\tilde{\mathbf{\Omega}}_{SUR} &= \begin{bmatrix} g_1\mathbf{I}_T & \begin{bmatrix} \mathbf{0}_{T-M} & g_1\mathbf{I}_{T-M} \end{bmatrix}' \\ \begin{bmatrix} \mathbf{0}_{T-M} & g_1\mathbf{I}_{T-M} \end{bmatrix} & g_2\mathbf{I}_{T-M} \end{bmatrix} \\ &= \begin{bmatrix} g_1\mathbf{I}_M & \mathbf{0}'_{T-M} & \mathbf{0}'_{T-M} \\ \mathbf{0}_{T-M} & g_1\mathbf{I}_{T-M} & g_1\mathbf{I}_{T-M} \\ \mathbf{0}_{T-M} & g_1\mathbf{I}_{T-M} & g_2\mathbf{I}_{T-M} \end{bmatrix},\end{aligned}$$

so that the corresponding matrix $\tilde{\mathbf{A}}$ is given by

$$\left(\tilde{\mathbf{X}}'\tilde{\mathbf{\Omega}}_{\mathbf{SUR}}^{-1}\tilde{\mathbf{X}}'\right)^{-1}\tilde{\mathbf{X}}'\tilde{\mathbf{\Omega}}_{SUR}^{-1} = \begin{bmatrix} \frac{1}{T}\mathbf{1}'_M & \frac{1}{T}\mathbf{1}'_{T-M} & \mathbf{0}'_{T-M} \\ \frac{1}{T}\mathbf{1}'_M & -\frac{M}{T(T-M)}\mathbf{1}'_{T-M} & \frac{1}{T-M}\mathbf{1}'_{T-M} \end{bmatrix}$$

which neither depends on g_1 nor on g_2.

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