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Threshold dynamics of short-term interest rates: empirical evidence and implications for the term structure

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Discussion Paper
Series 1: Economic Studies
No 02/2007

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Internet http://www.bundesbank.de

ISBN 978-3–86558–265–2 (Printversion)
ISBN 978-3–86558–266–9 (Internetversion)

Abstract:
This paper studies a nonlinear one-factor term structure model in discrete time. The single factor is the short-term interest rate, which is modeled as a self-exciting threshold autoregressive (SETAR) process. Our specification allows for shifts in the intercept and the variance. The process is stationary but mimics the nearly $I(1)$ dynamics typically encountered with interest rates. In comparison with a linear model, we find empirical evidence in favor of the threshold model for Germany and the US. Based on the estimated short-rate dynamics we derive the implied arbitrage-free term structure of interest rates. Since analytical solutions are not feasible, bond prices are computed by means of Monte Carlo integration. The resulting term structure exhibits properties that are qualitatively similar to those observed in the data and which cannot be captured by the linear Gaussian one-factor model. In particular, our model captures the nonlinear relation between long rates and the short rate found in the data.

Keywords:
Non-affine term structure models, SETAR models, Asset pricing

JEL-Classification:
E43, G12, C22

Non-technical summary

Dynamic models of the term structure of interest rates capture the joint dynamics of bond yields of different maturities. In the finance literature, the absence of arbitrage opportunities constitutes the key assumption that ties the movements of short-term and long-term interest rates together. Heuristically speaking, the no-arbitrage assumption rules out investment strategies that allow for riskless profits with a net-capital input of zero. Among the arbitrage-free approaches, the class of affine term structure models has become particularly prominent. In these models, arbitrage-free bond yields result as affine (linear plus an intercept) functions of a set of explanatory factors. This linear structure follows inter alia from the fact that the evolution of the factors themselves is assumed to be a linear autoregressive process. In the simplest example - the prototypical one-factor Vasiçek model - the short-term interest rate, depending linearly on its own past only, is the single driving force of the whole term structure.

However, the econometric literature provides evidence that the evolution of short-term interest rates is better described by nonlinear models in which, e.g., the law of motion depends on the level of the short rate itself. This paper estimates such a nonlinear specification of the short rate evolution for Germany and the US. The model allows for regime-shifts in the equilibrium level and the volatility of the one-month interest rate. Which regime prevails depends on whether the short rate has exceeded certain thresholds in a previous period. Thus, in the simplest case with only one threshold, the model distinguishes between high- and low-interest-rate regimes. For both countries, the specification with regime shifts turns out to be an adequate representation of the empirical dynamics, and statistical tests prefer this nonlinear formulation to linear models.

As a second contribution, the paper derives the arbitrage-free term structure based on the considered threshold process of the short rate. In contrast to the aforementioned affine models, analytical solutions for arbitrage-free bond prices do not exist here. Hence, they are computed by means of simulation methods (Monte Carlo integration). The resulting term structure exhibits properties that are qualitatively similar to those observed in the data and which cannot be captured by affine one-factor models. For instance, the threshold model delivers autocorrelations of interest rates that increase with maturity, and contemporaneous correlations between bond yields that decrease the more apart the maturities of the two yields, both features being in line with the data. In contrast, one-factor affine models imply constant autocorrelations across the maturity spectrum and contemporaneous correlations which are equal to unity for all maturities. Moreover, the threshold models exhibit a convex-concave relationship between the one-month rate and longer-term bond yields. This feature is also prevalent in the data and cannot be replicated by the affine Vasiçek model, which implies a linear relation for all maturities.

Nicht-technische Zusammenfassung

Dynamische Modelle der Zinsfristigkeitsstruktur erklären die gemeinsame zeitliche Entwicklung von Anleiherenditen verschiedener Laufzeiten. Dabei stellt die aus der finanzwirtschaftlichen Literatur stammende Annahme der Arbitragefreiheit das zentrale Bindeglied zwischen kurz- und langfristigen Renditen dar. Vereinfacht gesagt werden durch diese Bedingung solche dynamische Anlagestrategien ausgeschlossen, die risikolose Gewinne mit einem Kapitaleinsatz von Null ermöglichen würden. Eine wichtige Untergruppe der arbitragefreien Ansätze bilden die sogenannten affinen Modelle. Diese Zinsstrukturmodelle zeichnen sich dadurch aus, dass sich arbitragefreie Renditen aller Laufzeiten als affine (linear plus Absolutglied) Funktionen der Bestimmungsfaktoren ergeben. Diese lineare Struktur folgt unter anderem aus der Annahme, dass die Bestimmungsfaktoren selbst einem linearen autoregressiven Prozess folgen. Im einfachsten Beispiel - dem prototypischen Einfaktormodell von Vasiçek - stellt der kurzfristige Zinssatz, der wiederum ausschließlich (linear) von seiner eigenen Vergangenheit abhängt, die einzige Erklärungsgröße der gesamten Zinsstruktur dar.

Allerdings liefert die ökonometrische Literatur zahlreiche Hinweise darauf, dass die Entwicklung kurzfristiger Zinsen adäquater durch nichtlineare Modelle beschrieben werden kann, bei denen sich zum Beispiel das Entwicklungsgesetz des Zinses mit dessen Niveau verändert. Im vorliegenden Papier wird eine solche nichtlineare Spezifikation für die Entwicklung des kurzfristigen Zinses für Deutschland und die USA geschätzt. Das zugrundeliegende Modell sieht vor, dass das mittlere Niveau sowie die Volatilität des Einmonatszinses zwischen verschiedenen 'Regimen' hin- und herspringen können. Welches Regime zu einem bestimmten Zeitpunkt vorliegt, hängt davon ab, ob der Kurzfristzins im vorherigen Monat einen bestimmten Schwellenwert überschritten hat. Im einfachsten Fall, bei dem nur ein Schwellenwert berücksichtigt wird, unterscheidet der Modellierungsansatz zwischen Hoch- und Niedrigzinsphasen. Für beide untersuchten Länder stellt sich heraus, dass das Modell mit Regimewechseln eine adäquate Beschreibung der empirischen Zinsdynamik darstellt. In statistischen Tests wird dieses nichtlineare Modell der linearen Variante vorgezogen.

Basierend auf dem Schwellenwertprozess des Kurzfristzinses wird - als zweiter Beitrag des Papiers - die arbitragefreie Zinsstruktur abgeleitet. Im Gegensatz zu den oben angesprochenen affinen Modellen ergeben sich in diesem Fall keine analytischen Lösungen für Anleihepreise als Funktion des Kurzfristzinses. Sie werden deshalb mit Hilfe von Simulationsmethoden (sog. Monte-Carlo-Integration) berechnet. Die Charakteristika der resultierenden Zinsstruktur entsprechen qualitativ den in den Daten beobachteten, wobei es affinen Einfaktormodellen nicht gelingt, diese Eigenschaften abzubilden. So impliziert das Schwellenwertmodell zum Beispiel, dass Autokorrelationen von Zinsen mit der Laufzeit ansteigen und dass Korrelationen zwischen Zinsen verschiedener Laufzeiten umso kleiner werden, je größer die Differenz zwischen diesen Laufzeiten ist. Beide Ergebnisse stehen im Einklang mit den Eigenschaften der empirischen Daten. Im Gegensatz dazu implizieren affine Modelle Autokorrelationen, die über das Laufzeitenspektrum hinweg konstant sind, und kontemporäre Korrelationen, die für alle Paare von Laufzeiten den Wert eins annehmen. Darüber hinaus ergibt sich bei den Schwellenwertmodellen eine konvex-konkave Beziehung zwischen dem Einmonatszins und Renditen längerfristiger Anleihen. Dieses Muster findet sich ebenfalls in den Daten, es kann aber nicht mit dem affinen Vasiçek-Modell generiert werden; letzteres liefert vielmehr einen linearen Zusammenhang für alle Laufzeiten.

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Threshold Dynamics of Short-Term Interest Rates: Empirical Evidence and Implications for the Term Structure[1]

1 Introduction

The specification and estimation of time series models for the short-term interest rate has been the subject of numerous studies in the empirical finance literature. Besides being of independent interest, the dynamics of the short rate is the central input to models of the term structure of interest rates. In these models, the short rate is mostly assumed to exhibit the property of mean reversion. In contrast, most of the empirical work finds it hard to reject the hypothesis that the short rate is an integrated process.[2] Other studies, however, argue that the observed random-walk-like behavior may be attributed to omitted nonlinearity of the mean and volatility function.[3] Models with regime-switching are particularly prominent candidates for capturing these nonlinearities parametrically.[4]

By contrast, arbitrage-free term structure models are usually based on linear, mean-reverting processes of the short rate. In the popular class of affine multifactor models, the short rate is driven by a small number of linear state processes.[5] Generalizing the expectations hypothesis, arbitrage-free yields are risk-neutral expectations of average future short rates. Within the affine class, bond yields of all

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[2]In the econometrics literature it is generally agreed that interest rates and especially short-term interest rates contain a unit root. See, e.g., Rose (1988), Stock and Watson (1988), Enders and Siklos (2001), and Hansen and Seo (2002). See also Chan, Karolyi, Longstaff and Sanders (1992) who have estimated eight popular linear time series models used to specify interest rate dynamics, and have shown that there is only weak evidence for mean reversion.

[3]See, e.g., Lanne and Saikkonen (2002), Seo (2003), and Jones (2003).

[4]See, e.g., Gray (1996), Ang and Bekaert (2002a), Ang and Bekaert (2002b) and also the references given therein.

[5]See Piazzesi (2005) for an overview.

maturities are linear functions of the state or factor vector, where the 'intercept' and 'slope' coefficients depend on the parameters governing short rate dynamics and the parameters governing the market prices of factor-innovation risk. If there is only one factor in these models, it is the short-term interest rate itself as in the seminal studies of Vasicek (1977) and Cox, Ingersoll and Ross (1985).

This paper is positioned between the two strands of the literature, i.e. empirically motivated papers that provide evidence for nonlinear or nonstationary time series behavior of interest rates on one side, and the no-arbitrage term structure approach from the finance literature on the other side. It has two objectives: first we explore whether models from the class of self-exciting threshold autoregressive (SETAR) models proposed by Lanne and Saikkonen (2002) are an adequate representation of US and German interest rate behavior. The specification allows for shifts in the intercept and the variance. The process is stationary but due to stochastic level shifts mimics the nearly $I(1)$ dynamics typically encountered with interest rates. Second, we analyze the implications for the term structure that arise when one leaves the world of affine models and uses the estimated SETAR models as inputs for constructing the arbitrage-free yield curve. For this, we take the estimated parameters for the threshold process as given, we calibrate the market price of risk and obtain arbitrage-free bond yields using Monte Carlo methods.

We find that tests tend to prefer the threshold models over their linear counterparts. Concerning the term-structure implications, SETAR dynamics of the short rate imply that long-term bond yields are an S-shaped function of the short rate, a pattern that also seems to prevail empirically for the US. Moreover, our nonlinear one-factor term structure model is able to qualitatively capture certain stylized facts in the cross-sectional and dynamical pattern of US bond yields that the linear one-factor model is not able to generate.

Earlier related literature usually employed a version of the expectations hypothesis when linking longer-term bond yields to regime-switching short-rate processes. Examples are given by Hamilton (1988) and Kugler (1996) that employ a Markov-switching framework, or Pfann, Schotman and Tschernig (1996) that also use a SETAR model. An arbitrage-free approach is employed within the Markov-switching models of Bansal and Zhou (2002) and Dai, Singleton and Yang (2003) that allow for state-dependent transition probabilities. The no-arbitrage regime-switching model by Ang, Bekaert and Wei (2006) simultaneously captures the nominal as well as the real yield curve. The papers that are most closely related to our own study[6] are

[6]See also the continuous-time SETAR model by Decamp, Goovaerts and Schoutens (2004).

Gospodinov (2005) with a (possibly nonstationary) TAR-GARCH model, and Audrino and Giorgi (2005) who use discrete beta-distributed regime shifts constructed on multiple thresholds.

The remainder of the paper is organized as follows. In section 2, we focus on the time series behavior of interest rate dynamics, that is, we introduce the threshold process and discuss a test for nonlinearity within our framework. Section 3 presents the empirical results for US and German interest rate data. Section 4 discusses the implications of the threshold process for the term structure within the no-arbitrage framework. Finally, section 5 summarizes the results and provides an outlook to possible extensions of our analysis.

2 A Threshold Model for the Short-Term Interest Rate

2.1 Model Specification

For the dynamics of the short-term interest rate x_t, we consider the family of self-exciting threshold autoregressive (SETAR) processes proposed by Lanne and Saikkonen (2002).[7] which is given by

$$x_t = \nu + \sum_{k=1}^{m} \beta_k I(x_{t-d} \geq c_k) + \sum_{j=1}^{p} \phi_j x_{t-j} + \sigma(x_{t-d})\varepsilon_t, \tag{1}$$

where

$$\sigma(x_{t-d}) := \sigma + \sum_{k=1}^{m} \omega_k I(x_{t-d} \geq c_k),$$

$I(\cdot)$ is the indicator function and ε_t is a serially independent innovation with $E\varepsilon_t = 0$ and $E\varepsilon_t^2 = 1$. For the model parameters, we have $\beta_k \geq 0$ and $0 \leq c_1 < c_2 < \ldots < c_m$. The values of σ and $\omega_1, \ldots, \omega_m$ are such that $\sigma(x_{t-d})$ is always positive. The roots of the polynomial $\phi(L) = 1 - \sum_{j=1}^{p} \phi_j L^j$ are assumed to be outside the unit circle to guarantee geometric ergodicity for x_t. We will refer to the model as a SETAR(m, p, d), where m indicates the number of level shifts or regimes, p the lag order, and d the lag of the threshold variable. If we want to explicitly distinguish between the model with $\omega_k = 0$ for all k and the model with $\omega_k \neq 0$ for at least one k we will refer to the latter as the HSETAR (H standing for heteroskedasticity) specification.

[7] The following is based on the description in their paper to which the reader is referred for technical details.

When all β_k and all ω_k are zero, the model reduces to the standard linear and homoscedastic AR(p) model. Otherwise, the intercept and/or the variance are regime-dependent. There are $m+1$ different regimes, where the regime prevailing in period t depends on the value of the short rate at time $t-d$. For instance, for $m=1$, there are two regimes. If the threshold variable x_{t-d} is below the threshold value c_1, the process is in the lowest regime and behaves like a simple AR(p) model with a conditional long run mean of $\nu/(1-\phi_1-\phi_2-\cdots-\phi_p)$. For $x_{t-d} \geq c_1$ the process switches to a higher intercept $\nu+\beta_1$ and evolves around $(\nu+\beta_1)/(1-\phi_1-\phi_2-\cdots-\phi_p)$. The process switches back and forth between the different regimes with the same autoregressive parameters.

Although the process is stable, the stochastic level shifts generate nearly $I(1)$ dynamics typically encountered with interest rate behavior.[8] Heuristically, if the data are in fact from such a Data Generating Process (DGP), an econometrician who falsely ignores the level shift, would tend to overestimate the sum of autoregressive parameters and would tend to reject stationarity.

2.2 Estimation and Testing Approach

The estimation of the threshold process (1) is a straightforward application of Conditional Least Squares (CLS). In the case of the homoscedastic SETAR(m,p,d) model it consists of two steps. For given m,p,d and given threshold values $\bar{c}'=(\bar{c}_1,\ldots,\bar{c}_m)$, the parameter vector $\theta'=(\theta^{0\prime},\sigma)=(\nu,\beta',\phi',\sigma)$ with $\beta'=(\beta_1,\ldots,\beta_m)$ and $\phi'=(\phi_1,\ldots,\phi_p)$ can be estimated by ordinary least squares (OLS). We have

$$\hat{\theta}^0(\bar{c}) = \underset{\theta^0}{\operatorname{argmin}}\ S_T(\theta^0,\bar{c}), \quad \text{with } S_T(\theta^0,\bar{c}) = \sum_{t=t_0}^{T}(x_t - F(x_{t-1},\ldots,x_{t-p},x_{t-d};\theta^0))^2,$$

where $t_0=\max\{p,d\}+1$, the function $F(x_{t-1},\ldots,x_{t-p},x_{t-d};\theta^0)$ is the skeleton, i.e. the deterministic part of the model, and T indicates the sample size. The estimate $\hat{\sigma}(\bar{c})$ is obtained from the sum of squared residuals as usual. Thus, in the first step one obtains $\hat{\theta}(\bar{c})' = \left(\hat{\theta}^0(\bar{c})',\hat{\sigma}(\bar{c})\right)$.

In the second step, the vector $c'=(c_1,\ldots,c_m)$ of threshold values is determined by conducting a direct search over points in C, the set of allowable vectors c. For $m=1$, one considers the grid $X^g=\{x_{(0)},\ldots,x_{(T-d)}\}$ consisting of the order statistics of the observed data. The estimate of c_1 is obtained as

$$\hat{c}_1 = \underset{c_1\in X^g}{\operatorname{argmin}}\ S_T(\hat{\theta}(c_1),c_1). \tag{2}$$

[8] Another approach to discriminate regimes is to use an exogenous threshold variable or lagged differences as in Gospodinov (2005) or in Gonzalo and Montesinos (2002).

For the grid search, it has to be provided that each regime contains a pre-specified minimum fraction π^* of observations to produce reliable estimates, a popular choice is $\pi^* = 0.15$. For $m > 1$, an m-dimensional grid search has to be conducted. Finally, the CLS estimates of the parameters are given by $\hat{\theta} = \hat{\theta}(\hat{c})$.

The estimation of (1) with a regime-dependent error variance proceeds in the same manner. The difference is that the OLS estimator $\hat{\theta}$ in step one has to be replaced by a two-step weighted LS estimator with a heteroskedastic covariance matrix.

So far we have assumed that the order p of the lag-polynomial, the lag d of the threshold variable, and the number of thresholds m are known. In practice, the parameters p, d and m have to be determined on the basis of observed data as well. When we estimate models from the (H)SETAR family below, we follow Lanne and Saikkonen (2002) and consider lag lengths of $p = 1$ and 2, only. With respect to m, we use only values below or equal to 2 since this number is sufficient to remove the underlying level-shifting behavior from the data. For the delay parameter d of the threshold variable, we restrict ourselves to values between 1 and 3.[9]

Statistical inference for regime switching models suffers from the problem of unidentified nuisance parameters under the null hypothesis of no threshold, i.e. the linear model. The main consequence is that conventional statistical theory cannot be applied and analytical expressions of the limiting distribution of the test statistic do not exist. Critical values have to be determined by means of simulations. Furthermore, in our context, the high autocorrelation of the short rate dynamics implies that the process is $I(1)$ or nearly $I(1)$ under the null hypothesis. The unit root tests of Gonzalo and Gonzalez (1998) and Caner and Hansen (2001) for TAR models do not apply here, since their threshold variable is a stationary time series that differs from the regressand, whereas in our framework it is the lagged time series itself and possibly nonstationary under the null of $\beta_1 = \cdots = \beta_m = 0$. The approach by Lanne and Saikkonen (2002) circumvents this problem by putting the proposed model in the null hypothesis and conducting a stationarity test along the lines of Kwiatkowski, Phillips, Schmidt and Shin (1992). The following sketches this approach.

For the derivation of the test statistic, equation (1) is rewritten as follows:

$$x_t = \mu + \sum_{k=1}^{m} \phi(L)^{-1} \beta_k I(x_{t-d} \geq c_k) + z_t, \tag{3}$$

[9] Alternatively, the consistent estimation technique of the delay parameter d provided by Chan (1993) may be used.

where $\mu = \nu/\phi(L)$ and $\phi(L)z_t = \sigma(x_{t-d})\varepsilon_t$. This can be done since x_t is assumed to be a stationary process with stable roots in each regime and $\phi(L)$ is therefore invertible.

The null hypothesis of the test is "The DGP of x_t is a (H)SETAR model", i.e. $\beta_1, \ldots, \beta_m$ and $\omega_1, \ldots, \omega_m$ in the HSETAR are nonzero. It can be shown that under H_0, the series z_t is a stationary AR(p), whereas under the alternative it is assumed to be an unstable $I(1)$. Since z_t is unobservable, it is replaced by its empirical counterpart from a regression under the null hypothesis for constructing the test statistic.

The stationarity test is formulated in the framework of Leybourne and McCabe (1994) (L-M) and is based on a modified Lagrange-multiplier (LM) test statistic. After running the CLS regression of the (H)SETAR model and obtaining the residuals $\hat{\varepsilon}_t$, we compute

$$\hat{u}_t = \sum_{j=1}^{p} \hat{\phi}_j \hat{u}_{t-j} + \hat{\varepsilon}_t, \quad t = 1, \ldots, T \tag{4}$$

starting with $\hat{u}_t = 0$ for all $t \leq 0$. The $\hat{u}_t$ correspond to the standardized z-series, i.e. $u_t = z_t/\sigma$ or accordingly $u_t = z_t/\sigma(x_{t-d})$ using the regime-dependent residual variances in case of the heteroskedastic model. Since the LS estimates $\hat{\theta}$ lead to an inconsistent test, the following auxiliary ARMA(p,1) model has to be estimated using (quasi) Maximum Likelihood:[10]

$$\phi(L)\Delta\hat{u}_t = (1 - \psi L)a_t, \quad a_t \sim N(0,1) \tag{5}$$

to achieve proper estimates. In the next step, the sequence of $\hat{\varepsilon}_t$ is transformed back from $\hat{u}_t$ by using ϕ^*, the ML estimates of ϕ in equation (5):

$$\hat{e}_t^* = \hat{u}_t - \sum_{j=1}^{p} \phi_j^* \hat{u}_{t-j}.$$

The test statistic would employ a demeaned version of the residuals $\hat{e}_t^*$. Since such a statistic leads to a serious over-rejection when applied to strongly autocorrelated but stationary time series data Caner and Killian (2001), among others, proposed a modified version, which is also used in this paper. Consider therefore the following regression:

$$\hat{e}_t^* = \kappa - \gamma(L)\phi(L)\Delta\hat{u}_{t-1} + w_t + \gamma(L)w_{t-1}, \quad t = 2p+1, \ldots, T \tag{6}$$

with $\gamma(L) = \sum_{j=1}^{p} \gamma_j L^{j-1} = \sum_{j=1}^{p} \phi(1)^{-1}(\phi_j^* - \phi_j)L^{j-1}$, $\phi(L) = \sum_{j=0}^{p-1} \phi_j L^j$ with $\phi_j = \sum_{i=j+1}^{p} \phi_i$, $w_t = \sum_{j=1}^{t} \eta_j + \varepsilon_t$ and $\eta_t \sim \ iid(0, \sigma_\eta^2)$. When the null hypothesis

[10]Note that $\psi = 1$ under the null hypothesis.

holds, $w_t = \varepsilon_t$ and (6) becomes a constrained regression model with a moving average error component. For the (H)SETAR(1,1,1) we get $\gamma(L) = \gamma_1$, $\phi(L) = \phi_0$, and $\gamma(L)\phi(L) = \gamma_1\phi_0$ and equation (6) shrinks to

$$\hat{e}_t^* = \kappa - \gamma_1\phi_0\Delta\hat{u}_{t-1} + w_t + \gamma_1 w_{t-1}.$$

For the (H)SETAR(1,2,1) model, equation (6) can be written as

$$\hat{e}_t^* = \kappa - \gamma_1\phi_0\Delta\hat{u}_{t-1} - (\gamma_1\phi_1 + \gamma_2\phi_0)\Delta\hat{u}_{t-2} - \gamma_2\Delta\hat{u}_{t-3} + w_t + \gamma_1 w_{t-1} + \gamma_2 w_{t-2}.$$

It should be noted that if we take the restriction for γ_2 into account, we have to estimate five instead of six parameters by performing a constrained optimization.[11] Finally, under the null, the ML-estimated residuals of (6), say $\hat{\varepsilon}^*$, can be used to calculate the modified L-M statistic of our stability test:

$$Z(p) = \frac{\hat{\varepsilon}^{*\prime} V \hat{\varepsilon}^*}{(T-2p)\hat{\varepsilon}^{*\prime}\hat{\varepsilon}^*}, \tag{7}$$

where $\hat{\varepsilon}^* = (\hat{\varepsilon}^*_{2p+1} \ldots \hat{\varepsilon}^*_T)'$ and V is a matrix with ijth element equal to the minimum of i and j. Under the null hypothesis

$$Z(p) \xrightarrow{d} \int_0^1 [W(r) - rW(1)]^2 dr \tag{8}$$

where $W(r)$ is a standard Brownian motion and therefore $W(r) - rW(1)$ is a standard Brownian bridge. The same critical values apply as in the Kwiatkowski, Phillips, Schmidt and Shin (1992) unit root test. Since z_t contains no stochastic level shifts any more, its visual inspection together with x_t can be used to check the stationarity of z_t compared to x_t in addition to the formal test. Finally, model selection among estimated models with already stationary z_t series and different lag orders is conducting using the AIC and BIC information criteria adapted for the presence of the threshold effects.

In addition, we check whether the HSETAR specification (vs. the homoscedastic case) is required by testing for equality of the error variances across the different regimes. That is, we employ the LM-test for multiplicative heteroskedasticity within a standard linear model framework with stationary regressors along the lines of Harvey (1990), p. 170-172. This can be done since the threshold estimates are asymptotically independent of other estimates and we can treat them as true parameters values. Under the null hypothesis of $H_0 : \omega_1 = \cdots = \omega_m = 0$, the LM-statistic

$$LM^H = \frac{T(w'v)'(w'w)^{-1}w'v}{v'v},$$

[11] To estimate the parameters of the auxiliary regression of the model (for $p = 2$) involving multiplicative constraints, we cast it into a state space form and estimate it by Maximum Likelihood using the Kalman Filter algorithm.

where $w_t = (1\ I(x_{t-d} \geq c_1)\ \dots\ I(x_{t-d} \geq c_m))'$ and $v_t = \hat{\varepsilon}_t^2/\tilde{\sigma}_0^2 - 1$ is asymptotically χ_m^2 distributed. The parameter $\tilde{\sigma}_0^2$ is the ML estimate of $\sigma^2(x_{t-d})$ in equation (1).[12]

3 Estimation Results for US and German Interest Rates

For the following investigation we use short-term interest rates at monthly frequency for Germany and the US for the period January 1960 to December 2002. The German data are one-month (monthly average) money market rates reported by Frankfurt banks, whereas the US data are end-of-month estimates of continuously compounded zero-coupon government bond yields with a maturity of one month. We use average data for Germany since historical end-of-month data are available only as of January 1970.[13] For the analysis of term structure implications in the next section, the US data series is used together with yields for the maturities of 3, 6, 12, 24, 36, 48, 60, and 120 months. All government bond yields are smoothed Fama-Bliss data except of the ten-year yield which is taken from FRED (Federal Reserve Economics Data) and is a treasury constant maturity rate.[14] Figure 1 displays the US and German short-term interest rates.

A first glance confirms the high persistence of the short rate (see also Table 4) and indeed the augmented Dickey-Fuller (ADF) test does not reject the null of a unit root. As Rose (1988) argued, adding a deterministic linear trend to the Dickey-Fuller regression is not relevant for nominal interest rates, whereas adding an intercept retains the generality of the test. Since US and German short-term interest rates do not seem to follow a specific drift, we report here also the test statistics without intercept. Using an intercept (no intercept) and three lags to adjust for serial correlation, the value of the test statistic for the US is -2.423 (-1.124), whereas the 5% critical value is -2.868 (-1.940). For Germany, the statistic is -1.406 when no intercept is included in the ADF regression vs. -3.241 using an intercept. For both time series, the null hypothesis of an $I(1)$ process without intercept cannot be

[12]Note that σ is not restricted under the null.

[13]Monthly averages of interest rate data for Germany have been used, among others, by Cassola and Luis (2003). At the beginning of the sample, the data exhibit seasonal December spikes which are especially pronounced during 1960-1973. In order to avoid misleading results we have substituted those values, a total of 26, using interpolation.

[14]We thank Monika Piazzesi for making the data set (1 month to 60 months) for the US available on her website. The German data are obtained from the publicly available time series database at www.bundesbank.de.

rejected even at the 10% level.[15]

The estimation results of different specifications of the (H)SETAR model are presented in Table 1 for the US and in Table 2 for Germany. For both data sets, we present the results of two linear specifications, namely AR(1) and AR(2), the most simple threshold model, SETAR(1,1,1), and three additional threshold models. As mentioned in section 2.2, the additional models are chosen on the basis of information criteria, stationarity outcomes of the resulting z-series after removing the stochastic level shifts, and tests for heteroskedasticity. The estimated autoregressive parameter of the AR(1) and the sum of the parameters of the AR(2) are larger than 0.96 for the US and above 0.98 for Germany indicating (near) unit root behavior. The corresponding parameter values for the (H)SETAR models are considerably lower, e.g., 0.9176 in the case of the SETAR(2,2,1) for the US data set and 0.9322 for the HSETAR(2,2,1) for Germany. The sum of squared residuals (SSR) falls from 264.16 in the AR(1) and AR(2) models to less than 260 in several threshold specifications for the US sample denoting evidence for the level-shifting model, where the lowest value occurs for the HSETAR(2,2,3). The decline in the SSR-statistics for the German data is substantial only for the models with $p = 2$. Again, the lowest value belongs to the heteroskedastic model with $m = p = 2$. The stability test strongly suggests that removed level shifts account for nonstationarity. For the US, we obtain a value of 2.8578 for the AR(1) model, which is significant at the 1% level and much lower values for all threshold models. The $Z(p)$-statistics of the models with $m = 2$ indicate that adjusting for stochastic level shifts generates a stationary z_t-process since the null cannot be rejected at the 5% level (US) and 10% level (Germany), respectively.[16] The values of AIC and BIC tend to confirm these results as they are lowest for heteroskedastic specifications with two level shifts, i.e. HSETAR(2,2,3) for the US and HSETAR(2,2,1) for Germany.[17]

The parameters of the HSETAR models are highly significant in both data sets and the estimated σ-coefficients vary across the regimes. As found in the data, we get higher volatility values when the interest rates are high and smaller volatility values at lower levels of interest rates. Figures 2 and 3 plot the absolute changes of the

[15] See also the notes for tables 1 and 2.

[16] Our results are qualitatively similar to those obtained by Lanne and Saikkonen (2002) for the Swiss Franc three-month Eurorate during the period 01/1978 to 09/1997 and for UK Treasury Bill rates during the period 01/1964 to 09/1997.

[17] The information criteria for the threshold models are computed following Franses and van Dijk (2000), i.e. $\text{AIC}(p,m) = \sum_{k=1}^{m+1} T_k \ln \hat{\sigma}_k^2 + 2(p+m+1)$ and $\text{BIC}(p,m) = \sum_{k=1}^{m+1} T_k \ln \hat{\sigma}_k^2 + (p+1)\ln T_1 + \sum_{k=2}^{m+1} \ln T_k$, where T_k indicates the number of observations in each regime.

US and German data, respectively as a proxy for the underlying volatility together with the estimated σ-coefficients. The heteroskedastic specifications capture the changing volatility for the analyzed period quite well.

As outlined above, the visual inspection of the z-series in comparison with the original time series provides additional evidence that the nonstationarity is due to the estimated level shifts. The z-series of the threshold models show less trend behavior than the original time series or the z-series of the AR-processes. In particular, the z-series of the AR(2) in figure 4 reveals a potentially remaining trend or level-shifting behavior that is removed in the HSETAR(2,2,3) model. Again, similar results for the z-series apply to Germany, if we take a look at figure 5. The range of the HSETAR(2,2,1) is narrower than of the AR(2) or SETAR(1,1,1) specification.

4 Term Structure Implications

As is well known, arbitrage-free term structure models that are based on linear Gaussian short-rate processes imply a linear relationship between the short-term interest rate and longer-term bond yields. Thus, they belong to the popular class of affine factor models of Duffie and Kan (1996).[18] This section explores the implications for the term structure that arise from using the estimated nonlinear (H)SETAR model as the underlying state process. As common in the literature, we employ a stochastic discount factor approach which renders the resulting yield dynamics to be arbitrage-free.

We will first show how to map the short rate into bond prices and yields. The actual computations will employ Monte Carlo techniques, since the pricing equation is not given in closed form. After that we collect some stylized facts derived from our US term structure data. The last subsection derives the corresponding properties of the (H)SETAR models, and compares them to the features found in the data as well as to those implied by a linear one-factor model.

4.1 Methodology

We use a discrete-time framework as expounded in, e.g., Backus, Foresi and Telmer (1998). Consider a zero-coupon bond at time t with price P_t^n that has n periods left until maturity, where one period corresponds to one month. In the next period, this bond has only $n-1$ periods left until maturity and a price of P_{t+1}^{n-1}. It can

[18]See also Dai and Singleton (2000).

be shown that in an arbitrage-free bond market there exists a process $\{M_t\}$ with $E|M_t P_t^n| < \infty$ and $M_t > 0$ for all t, such that the following pricing equation holds

$$P_t^n = \mathrm{E}_t(M_{t+1} P_{t+1}^{n-1}), \tag{9}$$

where E_t represents the conditional expectations given information at time t. This fundamental pricing relation (9) is the core of a no-arbitrage term structure model. The random variable M_t is referred to as the stochastic discount factor (SDF) or pricing kernel. The pricing equation (9) can be iterated forward and the bond price can be written as the expected product of future pricing kernels,

$$P_t^n = \mathrm{E}_t\left(M_{t+1} M_{t+2} \dots M_{t+n}\right), \tag{10}$$

which uses the fact that $P_t^0 = 1$, i.e. the bond pays one unit of account at maturity. Apart from equation (9), every arbitrage-free term structure model consists of two additional components, a function g, $M_{t+1} = g(x_t, \varepsilon_{t+1})$, that specifies the dependence of the SDF on a state variable x and an innovation ε, and a law of motion of that state variable. Under continuous compounding, the yield y_t^n of a bond with n periods until maturity is obtained from the bond price as

$$y_t^n = -\frac{\ln P_t^n}{n}. \tag{11}$$

For the following analysis, the linear Gaussian one-factor model - that is, the discrete-time version of the prominent Vasicek (1977) model - will serve as a reference point for comparison. In this model, the short rate x_t is assumed to follow a stationary AR(1) process

$$x_t = \nu + \phi x_{t-1} + \sigma \varepsilon_t, \tag{12}$$

with $\varepsilon_t \sim i.i.d.\, N(0,1)$. In addition to that, we assume that the SDF satisfies

$$M_{t+1} = \exp\{-\delta - x_t - \lambda\sigma\varepsilon_{t+1}\}, \tag{13}$$

where $-\lambda$ is the market price of risk and $\delta = \lambda^2\sigma^2/2$. Using (12) and (13), the pricing equation (9) can be solved, that is, bond prices can be written as an explicit function of the state variable x_t. The solution is of the exponentially affine form,

$$P_t^n = \exp\{-A_n - B_n x_t\}. \tag{14}$$

The coefficients A_n and B_n depend on time to maturity but not on t. They satisfy the difference equations

$$B_n = \sum_{i=0}^{n-1} \phi^i = \frac{1-\phi^n}{1-\phi}, \tag{15}$$

$$A_n = \sum_{i=0}^{n-1} G(B_i), \tag{16}$$

with

$$G(B_i) = \delta + B_i\,\theta(1-\phi) - \frac{1}{2}(\lambda + B_i)^2\sigma^2$$

and initial condition $A_0 = B_0 = 0$. Using (11), bond yields are given as affine functions of the state variable

$$y_t^n = \frac{A_n}{n} + \frac{B_n}{n}x_t. \tag{17}$$

For $n = 1$, this solution delivers the short-term interest rate itself, $y_t^1 = x_t$.

Based on our threshold models, we consider two types of specifications for the corresponding term structure models. The first version is based on the short-rate process (1) with homoscedastic innovations, that is with $\sigma(x_{t-d}) = \sigma$. The functional form of the SDF is given by (13) with regime-independent market price of risk parameter λ. The second version allows for heteroskedasticity. For this variant, we follow Bansal and Zhou (2002) and allow for regime-dependent market prices of risk. The appropriate SDF of this heteroskedastic version is given by

$$M_{t+1} = \exp\{-\delta(x_{t-d+1}) - x_t - \lambda(x_{t-d+1})\sigma(x_{t-d+1})\varepsilon_{t+1}\}, \tag{18}$$

where $\lambda(x_{t-d}) = \lambda + \sum_{k=1}^{m} I(x_{t-d} \geq c_k)r_k$ for real constants $r_1, \ldots, r_m$.

For a given specification of the short rate evolution (and corresponding SDF formulation), we define the 'yield function' f_n as the mapping from the history of observed short rates $(x_t, x_{t-1}, x_{t-2}, \ldots)$ into the arbitrage-free yield y_t^n. We write

$$y_t^n = f_n(\mathcal{X}_t; \zeta). \tag{19}$$

where the vector ζ collects the parameters governing the time series process of the short rate θ as well as the parameter(s) λ (and r_m) controlling the market price of risk, and $\mathcal{X}_t$ denotes those past observations of the short rate, on which the conditional expectation (10) depends. For instance, if the underlying short-rate process is a linear AR(1), a SETAR(1,1,1) or a HSETAR(1,1,1), the conditional expectation in (10) - and thus any bond yield - will only depend on x_t and not on any past x_{t-i} , $i > 0$.[19] For the case of of a general HSETAR(m, p, d) with corresponding pricing kernel (18), the conditional expectation will be a function of $\mathcal{X}_t = (x_t, x_{t-1}, \ldots, x_{t-\max\{p,d\}+1})$.

For the term structure model based on the homoscedastic SETAR(1,1,1), an analytical expression for the yield function has been derived in Lemke and Archontakis (2006). It turns out that unlike with a linear short rate model, this function

[19]Hence, for the AR(1), $f_n(\mathcal{X}_t, \zeta) = f_n(x_t;\, \lambda, \sigma, \nu, \phi) = \frac{A_n}{n} + \frac{B_n}{n}x_t$.

is not affine anymore as the intercept does now depend on the current short rate in a nonlinear fashion,

$$y_t^n = \frac{A_n(x_t)}{n} + \frac{B_n}{n} x_t, \tag{20}$$

whereas B_n is the same as in equation (15), and again $y_t^1 = x_t$. For other processes from the HSETAR(m, p, d) family, an analytical solution will be more complicated, if it exists at all. Moreover, even for the simple SETAR(1,1,1), actually computing the yield function (20) suffers from a curse-of-dimensionality problem as explained in more detail in Lemke and Archontakis (2006). Hence, here we will make use of Monte-Carlo techniques to evaluate bond yields numerically.

For that, the conditional expectation on the right-hand side of (10), which is required to obtain the bond price P_t^n, is computed as follows. Conditional on the actual value x_t of the short rate and the respective lags $x_{t-1}, \ldots, x_{t-\max\{d,p\}+1}$ corresponding to the (H)SETAR model under consideration, one generates a realization of the sequence $\{x_{t+1}, \ldots, x_{t+n}\}$ using the (H)SETAR(m, p, d) model as the DGP. One then computes the corresponding sequence of pricing kernels $\{M_{t+1}, \ldots, M_{t+n}\}$ using (13) or (18), respectively. Then the product on the right hand side of (10) is computed and saved. The latter steps are repeated a large number of times, the average of the respective products is an estimate of the conditional expectation and thus of the bond price. This in turn can then be converted into a yield via (11). The yield function approximated in this manner will be denoted by $\hat{f}_n(\mathcal{X}_t; \zeta)$.

It turns out that in order to obtain sufficiently precise estimates of y_t^n (that are within a maximum error range of ± 0.01 percentage points for all times to maturity n) a large number of $N = 1,000,000$ simulated random paths of the SDFs is required. As can be derived from (11), the bond price P_t^n must not deviate from its theoretical value by more than -/+ $n/1200$ % if yields should be approximated with the desired precision.[20] The MC simulation is performed using antithetic variates to reduce the variance of the estimates. Essentially, when drawing innovations ε_t for the (H)SETAR processes, we generate a sample of $M/2$ *i.i.d.* normally distributed random variates and raise the sample size to a total of M variates by using the negative values of the realizations as well.

With the exception of the market price of risk λ, all parameters required for the computation of bond yields have already been estimated from time series data on the short rate in the previous section. We follow Backus, Foresi and Telmer (1998) and calibrate λ such that the actual average ten-year yield of the analyzed sample

[20]For instance, in order to have an $n = 12$-month yield of 3% p.a. within the interval [2.99, 3.01], the simulated bond price has to be within the narrow interval [0.97035,0.97045].

is matched exactly while the estimated parameters of the state process from section 3 are taken as given. In the case of regime-dependent risk parameters, we calibrate the m distinct λ-parameters in a similar fashion, using a higher-dimensional grid search. Since the model implies that $y_t^1 = x_t$, we have two exactly matched points in the average yield curve, the short rate and the ten-year yield.[21] The values of the λ-parameters resulting from the matching procedure are reported in Table 3, whereas the corresponding sample mean yield curves are not displayed here.

4.2 Stylized Facts

Before presenting the results from the simulation study, we focus on some important stylized facts regarding US bond yields for the period 01/1960 to 12/2002. Yields of all maturities are constructed using the same method (smoothed Fama-Bliss). Since such long time series are not readily available for Germany, we confine ourselves to the US data set.

Summary statistics of yields in levels for this period are reported in Table 4. Two important facts can be derived from this table. First, the data reveal the typical concave and upward sloping behavior in the sample mean yield curve. The average spread between the one-year and the one-month yield is 0.84 percentage points, the five-year-one-year spread amounts to 0.57, and the ten-year-five-year spread is 0.29. Second, the sample autocorrelation is very high, above 0.95 for all maturities and increasing with maturity. The same statistics for first differences are reported in Table 5.

The standard deviations of yield changes, sometimes referred to as the term structure of volatility, display a downward-sloping behavior with a sharper decline in volatility from the one-month bond to the three-month bond than in the remaining consecutive bonds. From six- to twelve-month bonds the volatility is even increasing.[22] Another characteristic feature of the relationship of different bond yields is that the contemporaneous correlation is high across the maturity spectrum, and

[21] Another possibility to obtain values for λ would be via an estimation of the complete parameter vector $(\nu, \beta_1, \ldots, \beta_m, c_1, \ldots, c_k, \sigma, \omega_1, \ldots, \omega_m, \phi_1, \ldots, \phi_p, \lambda)$ using additional cross-sectional information of the yield curve and simulation-based estimation methods, such as Simulated Maximum Likelihood (SML), or Simulated Method of Moments (SMM). In this paper, we restrict ourselves to the calibration method since it is sufficient for our objective to show the basic implications of a threshold effect for the term structure.

[22] This characteristic behavior of the curve is more pronounced, e.g., during the Greenspan era (08/1987 to 12/2005). Piazzesi (2005) describes the curve for that period as 'snake-shaped': high for short maturities ($<$ six months), low at six months, then increasing with a peak at two to three years, and then again decreasing.

decreases with the difference between maturities, see Table 6.

The last fact that needs to be pointed out for the subsequent analysis characterizes the relationship between long- and short-term yields and is displayed in figure 6. The graphs show plots of different n-period bond yields and the one-month short rate. A second-order penalized spline (P-spline) with 30 equidistant knots is fitted through the respective scatter plots.[23]

In general, the dispersion around the spline curve tends to increase with time to maturity. Moreover, the degree of nonlinearity exhibited by the bivariate relation between long-term yields and the short rate also seems to become more distinct for longer maturities. In the upper left plot, the functional form can be approximated with a simple upward sloping linear function. The 36- to 120-month yield plots exhibit a form that is convex when the short rate is low and concave at higher levels of the short rate.[24] This convex/concave shape seems to amplify for longer times to maturity. Moreover, the concavity is stronger emphasized indicating that long-term yields are less sensitive to the short rate at higher levels.

4.3 Simulation Results

In this section, we study the term structure properties that arise when the short-rate process is from the considered (H)SETAR family. We will consider the SETAR(1,1,1), the HSETAR(1,1,1) and the HSETAR(2,2,3). The SETAR(1,1,1) can be interpreted as that threshold process from the (H)SETAR family which is 'closest' to the linear AR(1) model, i.e. the discrete-time counterpart of the Vasiçek model. Thus, it is interesting to explore what differences it implies for the properties of the term structure, when the only difference compared to the linear AR(1) is the endogenously switching intercept. By including the HSETAR(1,1,1) into our synopsis, we will be able to explore what impact the additionally shifting volatility exerts on the joint behavior of bond yields. Finally, the HSETAR(2,2,3) is chosen since it

[23]In P-spline smoothing the unknown functional form is approximated by a large number (30-200) of knots and basis functions respectively, which is then fitted to the data imposing a penalty against overfitting (see, e.g., (?)). This guarantees a smooth fit while retaining the basic structure of the functional relationship. An application to interest rates can be found, e.g., in Krivobokova, Kauermann and Archontakis (2006).

[24]A cubic polynomial reflects the same behavior in an even more pronounced way. The $t-$statistics of the corresponding third order parameters are in absolute values larger than 2 for all maturities and around 4 for $n = 48$, 60, and 120. Here, we choose the spline to let the procedure choose a functional form that is not predetermined by itself. As a robustness check, we consider also smoothing splines with different smoothing parameters. The basic convex/concave functional form is confirmed by all nonparametric methods.

turned out to be the most adequate representation of the time series properties of the short rate, as discussed above in section 3. The parameter values for the three models are our estimates for the US.

Before presenting the results, the important disclaimer is in order that all term structure models based on our SETAR short rate processes are one-factor models. That is, as the linear Vasicek model, the whole continuum of bond yields is driven by one innovation process only. As such, the SETAR-based models cannot be expected to fit the complete dynamics of the term structure. Thus, we do not at all claim that the one-factor SETAR models are true competitors to the now established two- or three-factor models from the affine class of Duffie and Kan (1996). It is rather the objective to explore how the implied term structure properties differ from those implied by the benchmark linear Vasicek model: we assess whether the supposed form of nonlinearity of the driving process can help to capture certain features of yield curve dynamics, which would require an additional factor if one was staying within the class of linear models.

In particular, the analysis will focus on the following aspects. We will consider the mean yield curve, the term structure of volatility as well as the contemporaneous correlations and the autocorrelations of bond yields corresponding to the three threshold models. Moreover, it will be explored what the threshold models imply for the short-rate dependence of money-market and longer-term bond yields. Parallel to that, we will also consider the consequences of erroneously using the linear one-factor (Vasicek) model, when the data are in fact generated by one of the three indicated threshold models.

Mean yield curve and term structure of volatility First, consider the expected yield curve and the term structure of volatility implied by the threshold models. They are defined as the expectation of yields, $\mathrm{E}y_t^n$, and the standard deviation of yield changes, $\sqrt{\mathrm{Var}(\Delta y_t^n)}$, respectively, each viewed as a function of time to maturity n. These quantities are functions of the parameters of the underlying (H)SETAR process of the short rate and the market price of risk parameters. For affine term structure models based on linear short-rate processes, they can be computed analytically. For our processes from the (H)SETAR family, however, we have to rely on Monte Carlo techniques. Given the estimated parameters, $\hat{\zeta}$, the expected yield curve is given by

$$\mathrm{E}y_t^n = \mathrm{E}f_n(\mathcal{X}_t; \hat{\zeta}), \tag{21}$$

where f_n is the yield function defined in (19). The second expectation in (21) is taken with respect to the unconditional distribution of $\mathcal{X}_t$. This will be approximated as

$$\mathrm{E}f_n(\mathcal{X}_t;\hat{\zeta}) \approx \frac{1}{M}\sum_{i=1}^{M}\hat{f}_n(\mathcal{X}_i;\hat{\zeta}) \tag{22}$$

where $M = 1,000$ and $\mathcal{X}_i$ are draws from the unconditional distribution of $\mathcal{X}_t = (x_t, x_{t-1}, \ldots, x_{t-\max\{d,p\}+1})$. Hence, $\mathcal{X}_t = x_t$ for the SETAR(1,1,1) and HSETAR(1,1,1), whereas $\mathcal{X}_t = (x_t, x_{t-1}, x_{t-2})$ for the HSETAR(2,2,3). One draw of the unconditional distribution of $\mathcal{X}_t$ is obtained by generating 1,000 realizations of the respective (H)SETAR process and discarding the first 999 (or 997, respectively) of that.[25] Similarly, the term structure of volatility is approximated as

$$\sqrt{\mathrm{Var}(\Delta y_t^n)} = \sqrt{\mathrm{Var}\left(\Delta f_n(\mathcal{X}_t;\hat{\zeta})\right)} \approx \sqrt{\frac{1}{M}\sum_{i=1}^{M}\left(\Delta\hat{f}_n(\mathcal{X}_i;\hat{\zeta}) - \frac{1}{M}\sum_{i=1}^{M}\Delta\hat{f}_n(\mathcal{X}_i;\hat{\zeta})\right)^2}. \tag{23}$$

The model-implied autocorrelations, $\mathrm{Corr}(y_t^n, y_{t-j}^n)$, and the contemporaneous correlations between bond yields, $\mathrm{Corr}(y_t^n, y_t^m)$, will be approximated in a similar fashion.

Figure 7 presents points on the expected yield curves (left column of panels) and volatility curves (right column of panels) for maturities $n = 1$, 3, 6, 12, 24, 36, 48, 60, 84 and 120 months as stars. Based on our estimated parameters, the mean yield curves look similar, whereas the range ($\mathrm{E}y_t^{120} - \mathrm{E}y_t^1$) and also the 'degree of concavity'[26] differs across models. If one falsely assumes the state process to be a linear Gaussian one (solid lines), the econometrician's estimates will nevertheless imply a mean yield curve which is similar, although for the HSETAR models as true DGPs, the misspecification will incur a slight over-estimation of concavity.

The volatility curves of linear and nonlinear models are similar only for the SETAR(1,1,1). By adding the heteroskedastic component, the threshold model starts to differ from the AR(1). This difference becomes more distinct for the heteroskedastic model with two level shifts. In contrast to the AR(1), the HSETAR(2,2,3) generates a higher curvature for lower maturities, which is also a feature that has been found in characterizing the stylized facts in the data.

Autocorrelations and contemporaneous correlations Table 7 contains the autocorrelations of bond yields for the threshold models. The autocorrelation in-

[25] The computation of the expectation in (22) is in fact a Monte Carlo within Monte Carlo since for each of the $M = 1,000$ draws of $\mathcal{X}_i$, the yield function $\hat{f}_n(\mathcal{X}_i;\hat{\zeta})$ has to be computed using $N = 1,000,000$ simulation steps as explained above.

[26] For instance, the ratio of $\mathrm{E}y_t^{120} - \mathrm{E}y_t^{60}$ to $\mathrm{E}y_t^{60} - \mathrm{E}y_t^{6}$ would serve as a rough measure for that.

creases with maturity. This result is due to the nonlinearity of the underlying factor and resembles the autocorrelation structure of a multifactor term structure model qualitatively. Clearly, the model does not produce as sharply increasing autocorrelations as observed in the data, but this depends on the estimated parameters. Note that in contrast, all linear one-factor models produce the same autocorrelation for all maturities. The range of the autocorrelation between 1- and 120-month yields rises by adding the heteroskedastic term to the SETAR(1,1,1) model. Compared to the models with a single shift in the intercept, the HSETAR(2,2,3) has a considerably larger range and also a higher level of autocorrelation for all maturities.

With our threshold models, we also obtain a contemporaneous correlation that decreases with maturity (Table 8). Again, the HSETAR(1,1,1) produces a stronger decline in correlations than the SETAR(1,1,1). In the HSETAR(2,2,3) model this effect is even more pronounced. For the linear one-factor Vasicek model, in contrast, all contemporaneous correlations are unity, which is a direct consequence of the fact that the model is linear and contains one source of innovation only. Our threshold specifications also feature one source of randomness only, but their nonlinearities give rise to a correlation and autocorrelation pattern that a model from the affine class could only achieve with more than one innovation, i.e. in that class it could only be generated by a multifactor specification.

Bivariate relations between short rate and y_t^n Figure 8 compares the yield functions – the relationships between short rate and longer-term bond yields – of the linear Gaussian AR(1) model to that of our simplest threshold term structure model based on the SETAR(1,1,1). Maturities of $n = 2, 12, 60$, and 120 months are presented.

Both models display a positive relationship between the short and long-term rates that is declining for higher maturities. Although the only difference compared to the linear model is the change in the intercept, the implied differences for the yield function are considerable. First consider the upper-left panel, which contains the model-implied relationship between the one-month and the two-month yield. Both yield functions seem to be virtually identical as they lie upon each other. However, the yield function of the SETAR(1,1,1) is a stepwise linear function with a discontinuity in the form of a jump from 5.5 to 5.7 per cent at $\hat{c}_1$. To make this behavior more visible we take a 'zoom' around the threshold point $\hat{c}_1$, which is presented in the first panel of figure 9. The yield function of the Gaussian model goes exactly through this jump, exhibiting a slightly higher slope to adjust for the jumping effect.

The 'degree of nonlinearity' increases with time to maturity. The yield functions in figure 8 show a convex shape to the left of the threshold value and a concave shape to the right of it. As found in the data (compare with figure 6) and can be seen from the second panel of figure 9, the concavity is more emphasized for the longer maturity. Surprisingly, the model produces different types of nonlinearity with just the same autoregressive parameter in each prevailing regime and is able to qualitatively capture the functional form observed in the data. Pfann, Schotman and Tschernig (1996) and Gospodinov (2005), for instance, have shown similar effects, but they use more complex models that additionally allow for regime-dependent autoregressive parameters.[27].

Figure 10 plots the nonparametric spline estimate of the data-implied yield function together with the linear and nonlinear parametric yield functions of the different models. The HSETAR(1,1,1) is omitted from the graph since it is very similar to the SETAR(1,1,1) specification. For the three-month yield, all models match the data-implied yield function quite well. Moving to higher maturities, the results change significantly. Both AR(1) and SETAR(1,1,1) cannot fit the data. A model generating a higher slope is needed in order to match the pattern exhibited in the data. The HSETAR(2,2,3) model with two shifts in the intercept comes very close to the observed yield function and winds itself around the data-implied yield function.

It is noteworthy that apart from the calibrated values of the market-price-of-risk parameters λ, the parameters leading to the documented behavior of longer-term yields are estimated solely from time series information on the short rate. Nevertheless, the correlation and autocorrelation pattern of bond yields, the form of the mean yield and volatility curve and the relationship between long and short-term yields are captured surprisingly well with the nonlinear one-factor model. This reflects that the particular nonlinearity incorporated in the time-series behavior of the short rate translates (via the no-arbitrage condition) into a cross-sectional behavior of yields with different maturities that does match the corresponding features in the data better than the linear model. However, in order to improve upon the quantitative fit to empirical data, one would presumably have to include an additional factor, or enrich the dynamic specification of the short rate further.

[27] Allowing for regime dependent autoregressive parameters within our framework leads to a similar yield function to the right of the threshold compared to a model where only the intercept switches. To the left of the threshold, surprisingly, the yield function remains linear.

5 Concluding Remarks

The aim of this paper was to estimate and analyze certain threshold dynamics of short-term interest rates and to explore the implications of these dynamics for the term structure of bond yields. For Germany and the US, we estimated (H)SETAR models as proposed by Lanne and Saikkonen (2002). Using the estimated models, we derived the arbitrage-free term structure based on a stochastic-discount-factor approach. Unlike with models from the popular affine class, bond yields could not be computed analytically for higher maturities, so we had to rely on simulation techniques.

Two insights have been obtained. First, the near-$I(1)$ dynamics of US and German one-month interest rates can be well captured by SETAR models with endogenously switching intercepts and their extension with regime-dependent volatility (HSETAR). Tests against the linear alternative yield results in favor of the nonlinear threshold models.

Second, term structure models based on the one-factor SETAR process can generate certain stylized facts of yield curve behavior which linear one-factor models cannot match. Rather, in the class of linear models one would have to add an additional factor to obtain these features. The threshold models deliver autocorrelations that rise with maturity, and contemporaneous correlations that decrease the more apart the maturities of the two yields, both features being in line with the data. In contrast, linear Gaussian models imply constant autocorrelations across the maturity spectrum and contemporaneous correlations which are equal to unity for all maturities. Moreover, the heteroskedastic specification allows for a more flexible shape of the term structure of volatility compared to the Vasicek model. Finally, our nonlinear models exhibit a convex-concave relationship for the yield function, i.e. the bivariate mapping from the one-month rate into longer-term bond yields. This feature is also prevalent in the data and cannot be replicated by the Vasicek model, which implies a linear relation for all maturities. Our favorite specification, the HSETAR(2,2,3) model with two level shifts, regime-dependent volatility and regime-dependent market price of risk, fits the form of the empirical yield function quite well.

Summing up, the main message is that the particular SETAR dynamics of the short rate translate - via the no arbitrage condition - 'correctly' into the dynamic and cross-sectional properties of the whole yield curve: important stylized facts in term structure data are captured at least qualitatively, and the linear one-factor model is improved upon. However, as the SETAR model is still driven by one stochastic innovation only, it cannot be expected to be a true competitor to multifactor models.

Future research may consist of one of the three following extensions. First, instead of using the two-step approach (estimating the dynamics, then calibrating market prices of risk), simultaneous estimation of the complete term structure model may be conducted using simulation based methods such as Simulated Maximum Likelihood (SML), or the Simulated Method of Moments (SMM). The second suggestion alludes to the threshold process itself: it may be worthwhile to explore richer parameterized models that allow, e.g., for regime-dependent autoregressive coefficients. Finally, the one-factor nonlinear model could be extended to a multifactor term structure model which may then be compared to workhorse multifactor models from the affine class.

A Tables

Table 1: Estimation results for US one-month rate.

	AR(1)	AR(2)	SETAR(1,1,1)	HSETAR(1,2,3)	SETAR(2,2,1)	HSETAR(2,2,3)
ν	0.1998	0.1781	0.3058	0.1793	0.2217	0.1498
	(0.0744)	(0.0750)	(0.0846)	(0.0527)	(0.0950)	(0.0416)
β_1			0.2603	0.2041	0.1462	0.1262
			(0.1002)	(0.1243)	(0.1030)	(0.0493)
β_2					0.4097	0.3870
					(0.1517)	(0.1524)
c_1			5.5296	7.8252	3.1152	3.2472
c_2					5.5296	7.8252
ϕ_1	0.9629	0.8806	0.9253	0.8432	0.8422	0.8397
	(0.0122)	(0.0439)	(0.0189)	(0.0303)	(0.0458)	(0.0303)
ϕ_2		0.0860		0.1178	0.0754	0.1058
		(0.0440)		(0.0314)	(0.0440)	(0.0322)
SAR		0.9665		0.9611	0.9176	0.9405
σ_1				0.51086		0.2933
σ_2				1.3850		0.5493
σ_3						1.3840
σ	0.7176	0.7162	0.7136		0.7119	
SSR	264.16	262.10	260.72	258.00	257.98	255.54
LM^H			8.48e-7		2.39e-6	
$Z(p)$	2.8578	1.9427	0.7657	0.6903	0.3834	0.4602
Obs in:						
Lower			62%	84%	15%	16%
Middle					46%	68%
Upper			38%	16%	39%	16%
AIC	-337.82	-337.17	-502.34	-521.55	-504.72	-558.38
BIC	-329.32	-324.44	-491.54	-506.95	-490.89	-544.95

The sample covers the period from 01/1960 to 12/2002 and contains 516 observations. Estimation is conducted by CLS, standard errors are in parentheses. SAR is the sum of the autoregressive coefficients and SSR the sum of squared residuals. LM^H is the LM-test (marginal significance) for equality of error variances across the regimes. $Z(p)$ is the test statistic of the stability test, where the 1%, 5%, and 10% critical values are 0.739, 0.463, and 0.347, respectively. The (no-constant) ADF(3)-statistic for the sample is (-1.124) -2.423, the 1%, 5%, and 10% critical values are (-2.570) -3.445, (-1.940) -2.868, and (-1.616) -2.570, respectively.

Table 2: Estimation results for German one-month rate.

	AR(1)	AR(2)	SETAR(1,1,1)	SETAR(1,2,1)	HSETAR(2,1,1)	HSETAR(2,2,1)
ν	0.0808	0.0961	0.1889	0.2110	0.2354	0.2664
	(0.0473)	(0.0469)	(0.0671)	(0.0665)	(0.0332)	(0.0329)
β_1			0.2013	0.2135	0.1274	0.1389
			(0.0889)	(0.0878)	(0.0314)	(0.0299)
β_2					0.3371	0.3707
					(0.0598)	(0.0594)
c_1			8.1600	8.1600	5.0400	4.960
c_2					8.1600	8.160
ϕ_1	0.9850	1.1514	0.9583	1.126	0.9398	1.1164
	(0.0078)	(0.0435)	(0.0141)	(0.0445)	(0.0086)	(0.0208)
ϕ_2		-0.1691		-0.1727		-0.1842
		(0.0435)		(0.0434)		(0.0196)
SAR		0.9823		0.9540		0.9322
σ_1					0.2725	0.2721
σ_2					0.5383	0.5162
σ_3					0.6643	0.6592
σ	0.4434	0.4379	0.4417	0.4358		
SSR	100.89	98.004	99.888	96.880	98.832	95.642
LM^H			0.0003	0.0002		
$Z(p)$	0.8313	0.8572	0.4897	0.4234	0.2889	0.3222
Obs in:						
Lower			80%	80%	57%	55%
Middle					23%	25%
Upper			20%	20%	20%	20%
AIC	-833.52	-844.79	-897.71	-907.93	-984.94	-985.39
BIC	-825.03	-836.31	-887.04	-893.24	-972.17	-968.96

The sample covers the period from 01/1960 to 12/2002 and contains 516 observations. Estimation is conducted by CLS, standard errors are in parentheses. SAR is the sum of the autoregressive coefficients and SSR the sum of squared residuals. LM^H is the LM-test (marginal significance) for equality of error variances across the regimes. $Z(p)$ is the test statistic of the stability test, where the 1%, 5%, and 10% critical values are 0.739, 0.463, and 0.347, respectively. The (no-constant) ADF(3)-statistic for the sample is (-1.406) -3.241, the 1%, 5%, and 10% critical values are (-2.570) -3.445, (-1.940) -2.868, and (-1.616) -2.570, respectively.

Table 3: Calibrated market price of risk parameters for the term structure models.

	AR(1)	SETAR(1,1,1)	HSETAR(1,1,1)	HSETAR(2,2,3)
λ	-210	-155		
λ_1			-260	-110
λ_2			-180	-180
λ_3				-100

Table 4: Summary statistics of monthly yields.

Mat n	Mean	St Dev	Skewness	Auto
1	5.496	2.586	1.223	0.9576
3	5.906	2.721	1.183	0.9772
6	6.108	2.749	1.133	0.9779
12	6.335	2.689	0.9916	0.9766
24	6.546	2.624	0.9557	0.9806
36	6.714	2.544	0.9619	0.9820
48	6.841	2.509	0.9467	0.9827
60	6.908	2.483	0.9156	0.9847
120	7.200	2.526	0.9506	0.9881

For each time to maturity (Mat) measured in months, the table contains mean, standard deviation, skewness, and first autocorrelation of the respective yields (continuously compounded) in percent p.a.

Table 5: Summary statistics of monthly yields in first differences.

Mat n	Mean	St Dev	Skewness	Auto
1	-0.0044	0.7233	-1.031	-0.1036
3	-0.0055	0.5370	-1.416	0.1308
6	-0.0065	0.5310	-1.573	0.1565
12	-0.0067	0.5316	-1.056	0.1211
24	-0.0059	0.4607	-0.7030	0.1647
36	-0.0052	0.4278	-0.1095	0.1220
48	-0.0043	0.4148	-0.1382	0.0701
60	-0.0036	0.3833	-0.2474	0.0847
120	-0.0013	0.3000	-0.4176	0.3185

Same as table 4 but for first difference of yields.

Table 6: Contemporaneous correlation of monthly yields.

Mat n	1	3	6	12	24	36	48	60	120
1	1.00								
3	0.987	1.00							
6	0.973	0.990	1.00						
12	0.966	0.986	0.987	1.00					
24	0.942	0.966	0.969	0.991	1.00				
36	0.920	0.946	0.950	0.977	0.996	1.00			
48	0.902	0.929	0.933	0.963	0.989	0.997	1.00		
60	0.888	0.916	0.920	0.951	0.982	0.994	0.998	1.00	
120	0.861	0.888	0.892	0.924	0.963	0.979	0.988	0.992	1.00

Table 7: Autocorrelation of yields in the threshold term structure models.

Mat n	SETAR(1,1,1)	HSETAR(1,1,1)	HSETAR(2,2,3)
1	0.96651	0.96998	0.97974
3	0.96797	0.97121	0.98254
6	0.96882	0.97197	0.98332
12	0.96953	0.97238	0.98593
24	0.97006	0.97240	0.98705
36	0.97025	0.97275	0.98877
48	0.97029	0.97288	0.98930
60	0.97034	0.97298	0.98932
84	0.97053	0.97297	0.98924
120	0.97042	0.97298	0.98933

Table 8: Correlation of yields in the threshold term structure models.

Mat n	1	3	6	12	24	36	48	60	84	120
					SETAR(1,1,1)					
1	1.000									
3	0.9997	1.000								
6	0.9988	0.9997	1.000							
12	0.9966	0.9982	0.9994	1.000						
24	0.9931	0.9955	0.9975	0.9993	1.000					
36	0.9911	0.9938	0.9962	0.9985	0.9998	1.000				
48	0.9899	0.9927	0.9953	0.9980	0.9996	0.9999	1.000			
60	0.9892	0.9921	0.9948	0.9976	0.9994	0.9999	0.9999	1.000		
84	0.9885	0.9915	0.9943	0.9973	0.9992	0.9998	0.9999	0.9999	1.000	
120	0.9881	0.9911	0.9940	0.9970	0.9991	0.9997	0.9998	0.9999	0.9999	1.000
					HSETAR(1,1,1)					
1	1.000									
3	0.9996	1.000								
6	0.9984	0.9995	1.000							
12	0.9958	0.9977	0.9992	1.000						
24	0.9918	0.9945	0.9970	0.9992	1.000					
36	0.9895	0.9925	0.9955	0.9982	0.9997	1.000				
48	0.9882	0.9914	0.9945	0.9976	0.9994	0.9999	1.000			
60	0.9874	0.9906	0.9939	0.9971	0.9992	0.9997	0.9999	1.000		
84	0.9864	0.9897	0.9931	0.9966	0.9989	0.9995	0.9997	0.9998	1.000	
120	0.9855	0.9889	0.9924	0.9959	0.9984	0.9991	0.9994	0.9995	0.9996	1.000
					HSETAR(2,2,3)					
1	1.000									
3	0.9996	1.000								
6	0.9986	0.9996	1.000							
12	0.9961	0.9979	0.9992	1.000						
24	0.9911	0.9937	0.9962	0.9988	1.000					
36	0.9872	0.9903	0.9934	0.9970	0.9996	1.000				
48	0.9845	0.9878	0.9913	0.9954	0.9988	0.9998	1.000			
60	0.9825	0.9860	0.9897	0.9942	0.9981	0.9995	0.9999	1.000		
84	0.9801	0.9837	0.9877	0.9926	0.9971	0.9988	0.9996	0.9999	1.000	
120	0.9786	0.9823	0.9864	0.9915	0.9963	0.9983	0.9992	0.9996	0.9999	1.000

B Figures

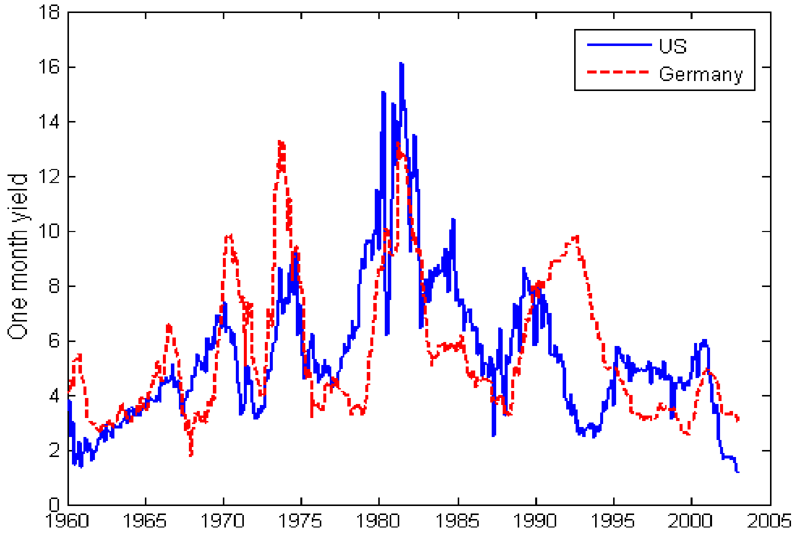

Figure 1: One-month interest rates for Germany and the US.

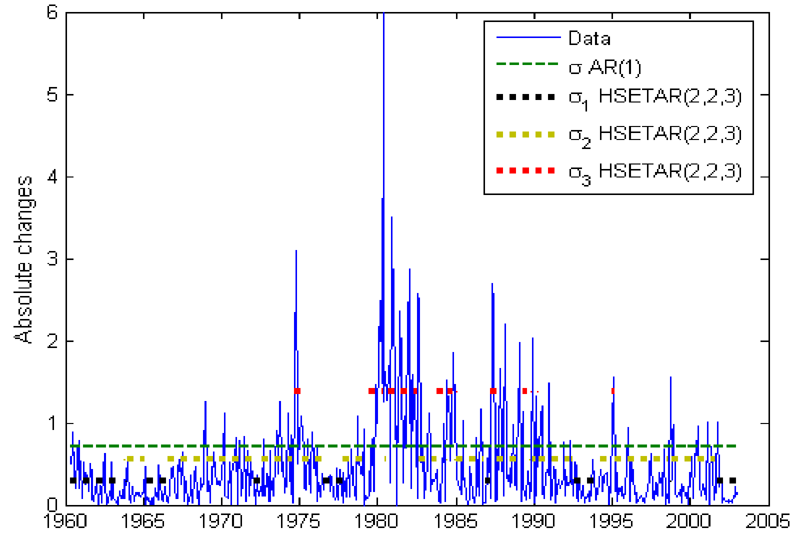

Figure 2: Absolute changes of US short rate.

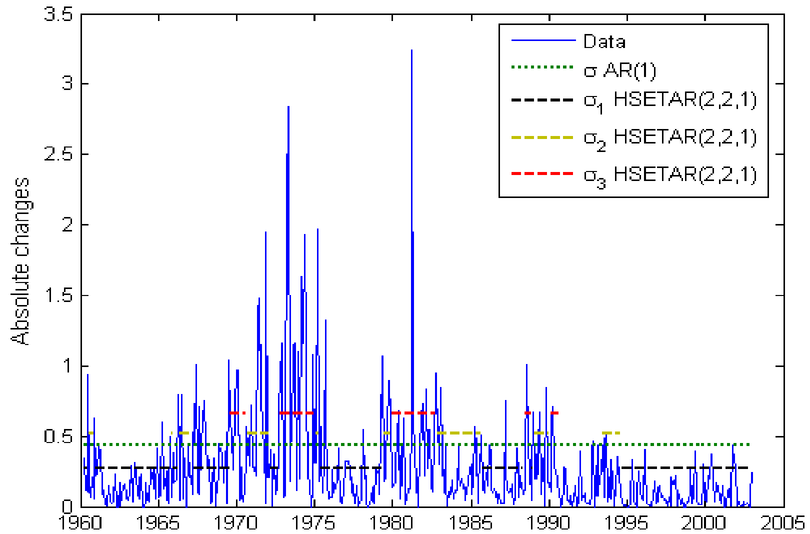

Figure 3: Absolute changes of German short rate.

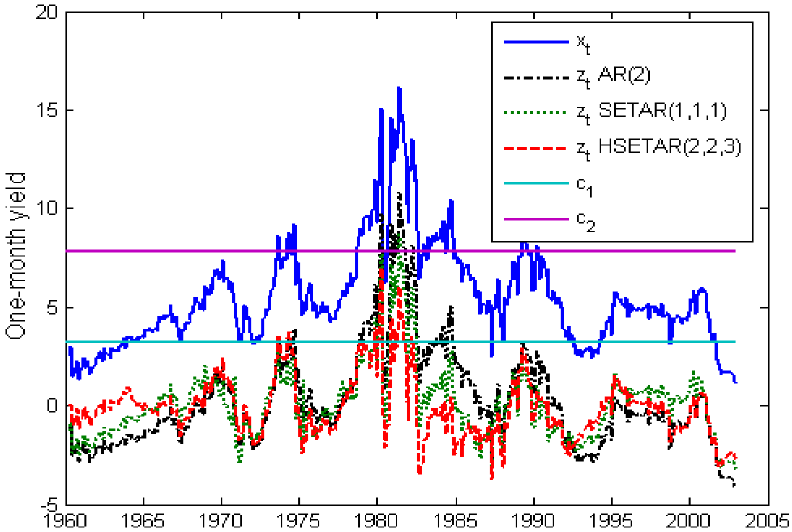

Figure 4: Estimated z-series of US short rate.

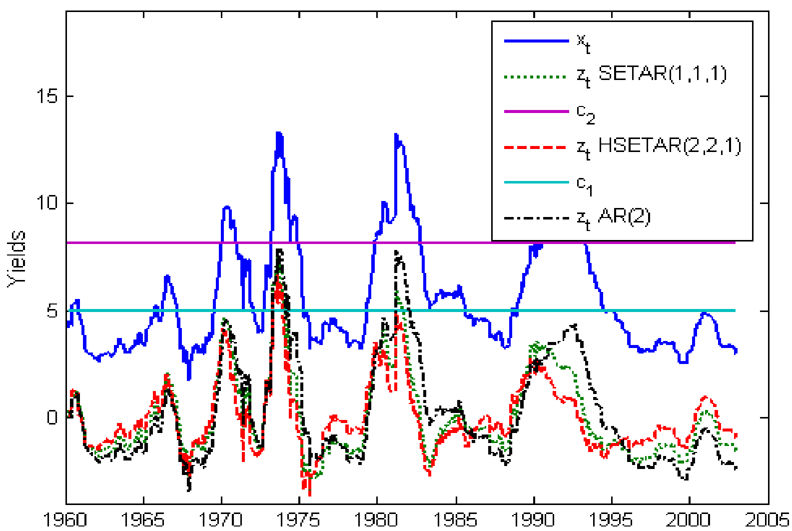

Figure 5: Estimated z-series of German short rate.

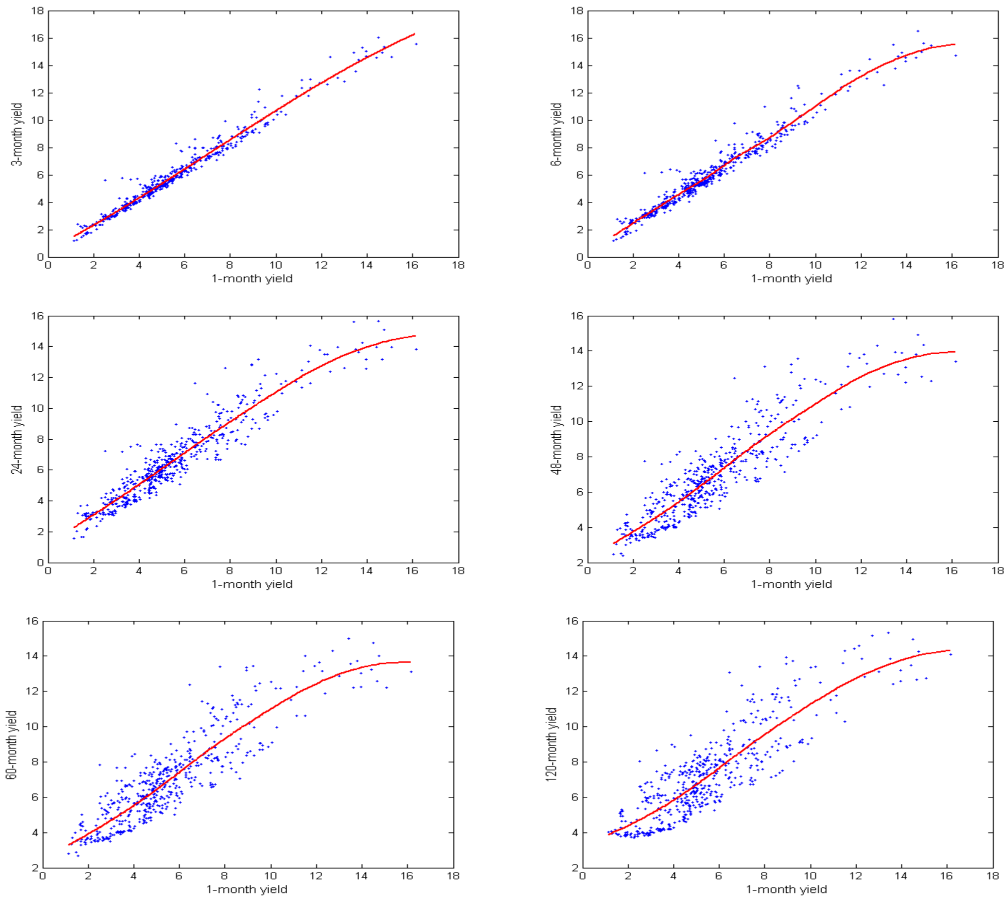

Figure 6: Plot of US long-term yields against the one-month short rate. The functional form of the underlying yield function is estimated via a nonparametric P-Spline.

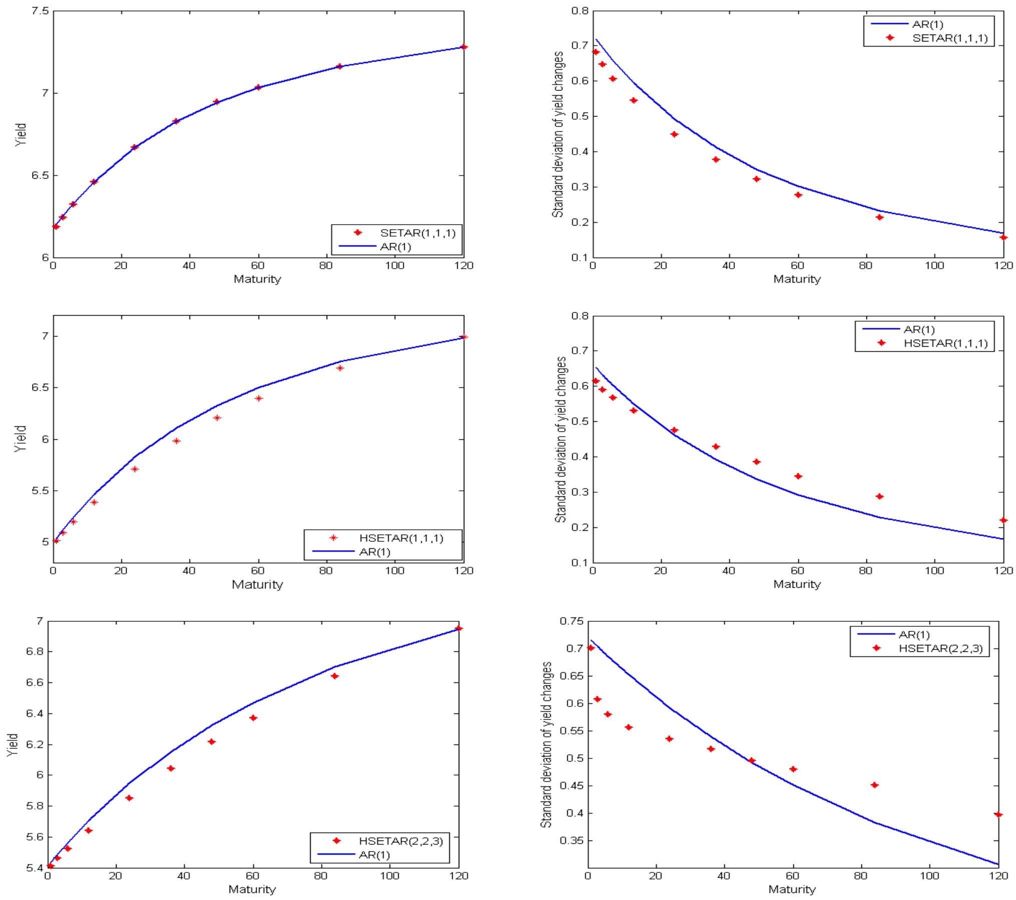

Figure 7: Mean yield curves and term structure of volatility for threshold and linear Gaussian model. The mean and volatility curves of the threshold models (SETAR(1,1,1), HSETAR(1,1,1) and HSETAR(2,2,3)) are based on sampled data. The Gaussian model is estimated using data generated from the corresponding threshold models, which are treated as the 'true' models.

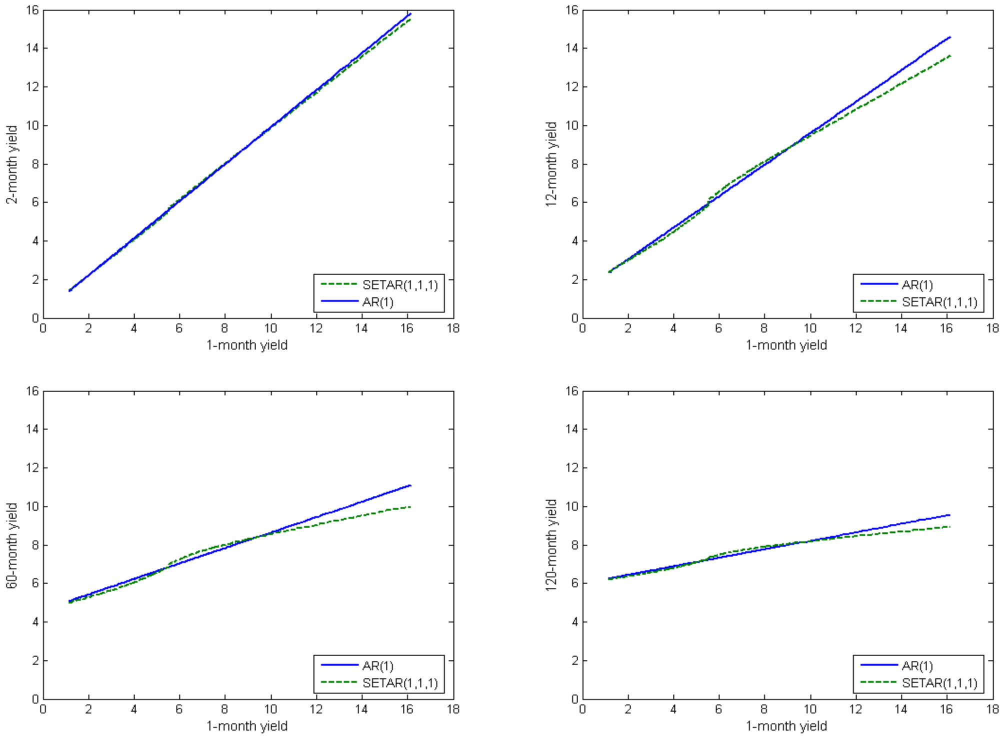

Figure 8: Implied n-period yields as functions of the short rate.

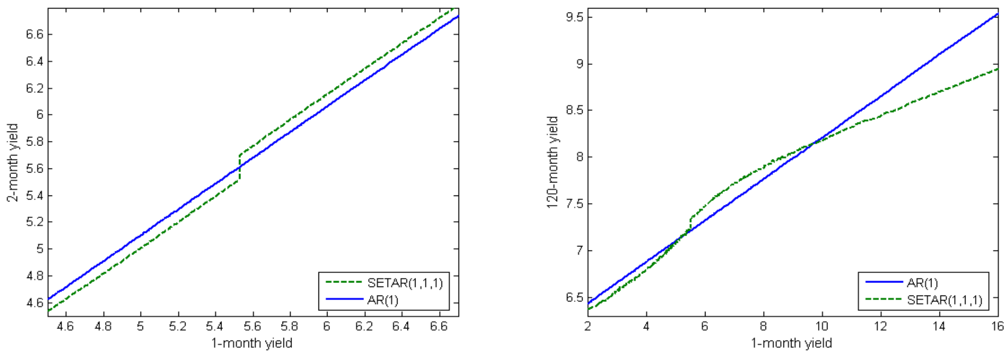

Figure 9: Implied 2- and 120-period yield as function of the short rate. Note the 'zoom' on the horizontal axis of the left panel.

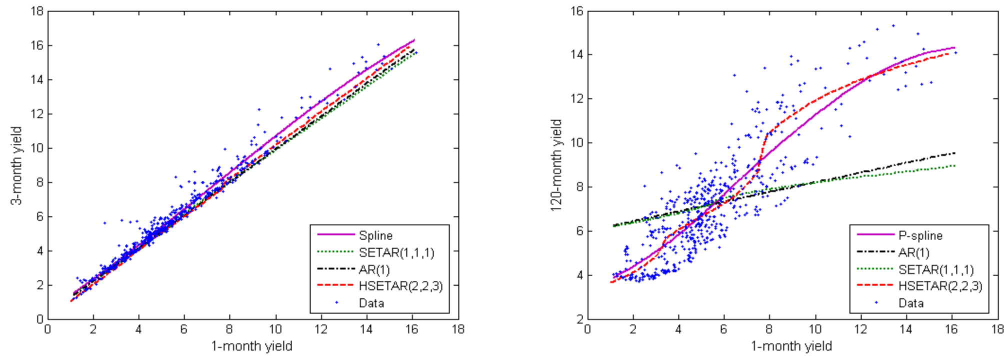

Figure 10: Data- and model-implied yield functions.

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