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Bond pricing when the short term interest rate follows a threshold process

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Abstract:

Using a stochastic discount factor approach, we derive the exact solution for arbitrage-free bond yields for the case that the short-term interest rate follows a threshold process with the intercept switching endogenously. The yield functions, mapping the one-month rate into n-period yields, exhibit a convex-concave shape to the left and the right of the threshold value, respectively. This is in contrast to linear short-rate processes which imply an affine yield function. The intervals for which convexity or concavity prevails increase with time to maturity.

JEL Classification: E43, G12, C63

Keywords: Threshold process, term structure of interest rates, nonlinear yield function

Non-Technical Summary

In dynamic factor models of the term structure, the joint evolution of interest rates of different maturities is ascribed to a small set of driving forces. If there is a single factor, it will usually coincide with the short-term (one-month) interest rate. The reaction of long-term yields to variations of the short rate is restricted by the condition of no arbitrage. Loosely speaking, the no-arbitrage assumption precludes trading strategies in bond portfolios that are characterized by zero initial net payments but guaranteed profits in the future. In no-arbitrage models it is mostly assumed that the factor process is linear and has Gaussian innovations. This setting implies that arbitrage-free bond yields are linear functions of the short rate: the sensitivity of long-term yields with respect to changes of the short rate is independent of the short rate's level.

In contrast, the empirical literature finds evidence that the dynamics of the short-term interest rate is characterized by nonlinearities, time-varying volatility and innovations which are not normally distributed. These deviations from linear Gaussian short rate models usually render an analytical solution for arbitrage-free bond yields infeasible.

This paper analyzes the term structure implications for such a nonlinear case. The short-term interest rate follows a threshold process, for which the intercept switches endogenously in an otherwise standard first-order autoregressive specification. As shown in the literature, this specification is especially suited to capture the near-random walk behavior of short-term interest rates. We derive the pricing function, that is the mapping between the one-month rate and n-period yields. The relationship between the short rate and any yield of longer maturity exhibits a convex-concave shape: the sensitivity of long-term yields with respect to changes in the short rate is first increasing in the level of the short rate but then decreases as the short-term interest rate increases further. This pattern is the more distinct the higher the maturity of the long-term bond.

Nicht technische Zusammenfassung

In dynamischen Faktormodellen der Zinsstruktur wird die gemeinsame zeitliche Entwicklung von Zinsen verschiedener Laufzeiten auf eine kleine Zahl von Bestimmungsgrößen zurückgeführt. In Modellen mit nur einem Faktor ist dies meist der kurzfristige Zins (Laufzeit ein Monat). Die Reaktion von langfristigen Renditen auf Veränderungen des Einmonatszinses wird durch die Bedingung der Arbitragefreiheit beschränkt. Das heißt, dass es keine Handelsstrategien gibt, die durch einen anfänglichen Nettokapitaleinsatz von Null, aber einen garantierten Gewinn in der Zukunft charakterisiert sind. In arbitragefreien Modellen wird meist unterstellt, dass der Faktorprozess linear ist und normalverteilte Innovationen aufweist. Unter diesen Annahmen lassen sich Anleiherenditen für alle Laufzeiten als lineare Funktionen des kurzfristigen Zinses darstellen. Das bedeutet, dass das Ausmaß der Reaktion von langfristigen Renditen auf Änderungen im Einmonatssatz unabhängig von dessen Niveau ist.

Die empirische Literatur liefert allerdings Belege dafür, dass die Dynamik des kurzfristigen Zinses durch Nichtlinearitäten, zeitvariierende Volatilität und Innovationen, die nicht normalverteilt sind, charakterisiert ist. Mit diesen Abweichungen vom linearen Gaußschen Modell ist es nur in wenigen Spezialfällen möglich, arbitragefreie Anleiherenditen analytisch zu berechnen.

Dieses Papier ermittelt die arbitragefreie Zinsstrukturdynamik für einen solchen Fall: Der kurzfristige Zins folgt einem autoregressiven Schwellenwert-Prozess, bei dem der Niveauparameter endogen zwischen zwei Werten hin und her wechselt. Wie in der Literatur gezeigt wird, ist diese Spezifikation besonders gut geeignet, das dynamische Verhalten des Kurzfristzinses, das oft dem eines Random Walk nahe kommt, zu beschreiben. Wir leiten analytisch die Preisfunktion, das heißt den funktionalen Zusammenhang zwischen Kurzfristzins und langfristigen Renditen, her. Diese Funktion weist eine konvex-konkave Gestalt auf: Langfristzinsen reagieren mit zunehmendem Niveau der Einmonatsrate zunächst stärker auf deren Änderungen, ab einem bestimmten Zinsniveau nimmt die Stärke der Zinsreaktion allerdings ab. Je länger die Laufzeit der betrachteten Anleihe ist, desto ausgeprägter ist dieses Muster.

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1 Introduction

Starting from the seminal contributions of Vasiček (1977) and Cox, Ingersoll, and Ross (1985), there is by now a large and growing literature that tries to explore the nature of the dynamics of interest rates and the relationship of yields with different maturities in an arbitrage-free framework. While models in the finance literature were preferably formulated in continuous time with continuous state space or in discrete time with discrete state space, models that are using discrete time and a continuous state space have become increasingly popular recently. This may in part be attributed to the fact that this framework is familiar to macroeconomists enabling them to integrate term structure elements in models of macroeconomic dynamics.

Models of the latter type (continuous state space, discrete time), usually consist of two components: a specification of the dynamics of the state vector and a formulation of the stochastic discount factor. Given these, the condition of no-arbitrage determines the dynamics of the whole spectrum of bond yields. If the state or factor vector is one-dimensional it usually coincides with the short-term (one-month) interest rate.

While the large empirical literature devoted to modeling and estimating short term interest rate dynamics is bringing up increasingly rich and advanced specifications, the literature on arbitrage-free term structure models usually restricts itself to simple linear state dynamics. This is because "researchers are inevitably confronted with trade-offs between the richness of econometric representations of the state variables and the computational burdens of pricing and estimation" as Dai and Singleton (2000) observe. They conclude that this is the reason why there is a huge emphasis on models from the affine

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class in the literature. Affine models are treated in a unified framework by Duffie and Kan (1996), their properties are further analyzed by Dai and Singleton (2000).[1]

Models of this class are characterized by a solution that expresses bond yields as an affine function of the state vector. This follows from linear state dynamics, Gaussian innovations and a stochastic discount factor that is a linear function of the state vector. If one of these assumptions is dropped, bond yields can generally not be expressed as affine functions of the state vector.[2] Moreover, there are only a few models outside the affine class that allow for an analytic solution at all: examples are the regime switching model by Bansal and Zhou (2002) and the quadratic model by Ahn, Dittmar, and Gallant (2002).

This paper contributes to the literature by deriving the analytical solution for bond yields for the case that the short rate follows a threshold process of the type presented by Lanne and Saikkonen (2002). Their formulation is especially suited to capture the near unit-root dynamics of interest rates. We consider the simplest version of their model in which the law of motion is an AR(1) with homoscedasitic Gaussian innovations. The intercept is allowed to change between two regimes. The regime prevailing is determined by the previous period's realization of the short rate, i.e. the model is of the SETAR (self exciting threshold autoregressive) type.

Papers that consider term structure implications of threshold dynamics usually do so by means of simulation.[3] However, Gospodinov (2005) remarks in a footnote that an analytical solution may be feasible for certain special cases of the fairly general TAR-GARCH model considered there.

Compared to an affine Gaussian one-factor model, the only difference of our state process is the changing intercept. However, it turns out that this slight modification induces substantial changes to the solution compared to the affine model. The yield function, mapping realizations of the short rate into yields of longer maturities, is nonlinear and exhibits a point of discontinuity at the threshold value. The function exhibits a convex-concave pattern, a phenomenon that qualitatively matches similar patterns observed in the data. For values of the short rate sufficiently far off the threshold value, however, the yield function is approximately linear. The width of the interval for which nonlinearity

[1] The discrete-time version of the affine class is described by Backus, Foresi, and Telmer (1998).

[2] See, however, Backus et al. (1998) and Lemke (2006) who show that replacing the normal distribution of innovations by a Gaussian mixture still leads to an affine yield function.

[3] See Pfann, Schotman, and Tschernig (1996) and Gospodinov (2005). The paper by Audrino and Giorgi (2005) can be seen as an exception. It features beta-distributed regime shifts and exhibits a similar structure as the Markov regime switching model in Bansal and Zhou (2002).

in the yield function prevails is increasing with time to maturity.

However, there is a problem with the derived exact yield function as it can actually be computed for yields with a small time to maturity only (say up to six months). This is because computing the function for the n-period yield requires the value of the cumulative distribution function of an $(n-2)-$dimensional normal with non-diagonal variance-covariance matrix. Moreover, the number of required computations increases exponentially with time to maturity, posing in addition a curse-of-dimensionality problem. Accordingly, for longer times to maturity, approximations of our exact solution or simulation-based techniques have to be applied.[4]

The structure of the paper is as follows. The next section gives a description of the model, section 3 derives the analytical yield function, followed by a numerical example in section 4. The fifth section concludes, an appendix contains a detailed derivation of the yield function.

2 The Model

The model that we consider is in discrete time and operates on a continuous state space. One unit of time may be thought of as one month. The single state variable in our one-factor model is the one-month interest rate. Its dynamics is given by the SETAR specification

$$X_t = \nu + \beta\, I(X_{t-1} \geq c) + \kappa\, X_{t-1} + \sigma\epsilon_t, \quad \epsilon_t \sim N(0,1) \tag{2.1}$$

where $I(\cdot)$ is the indicator function and the innovations ϵ_t are serially independent. The parameter κ is in the interval $(0,1)$, guaranteeing stationarity of the process. The only difference to a linear Gaussian model – i.e. the discrete-time version of the Vasiček model[5] – is the time-varying intercept. Depending on the previous realization of the short rate it is given by ν or $\nu+\beta$, respectively. Note that a regime-dependent intercept induces a regime-dependent long-run mean of the short rate. Approximating the data generating process of the one-month interest rate with a standard AR model (one intercept) usually requires a value of κ close to unity to capture the high persistence of the short rate process as observed in the data. Heuristically, the two-intercepts specification requires a lower κ since the short rate process is now allowed to revert to two different means.

[4]See Archontakis and Lemke (2005).
[5]See Backus et al. (1998).

The process given by (2.1) is the simplest member in a class of models proposed by Lanne and Saikkonen (2002). The more general specification is written as

$$X_t = \nu + \sum_{k=1}^{r} \beta_k \, I(X_{t-d} \geq c_k) + \sum_{j=1}^{p} \kappa_j X_{t-j} + \sigma(X_{t-d})\epsilon_t \tag{2.2}$$

i.e. it allows for more than two regimes, for more lags in the autoregressive specification, and for the threshold variable being lagged by more than one period. The coefficient of state innovations is also allowed to be regime-dependent, allowing for regime-dependent variance. In fact, the speciation that is most adequate empirically turns out to be heteroscedastic, with r, d, p all exceeding unity.[6] The reason for sticking to the case with $r = d = p = 1$ is that we want to point out the effects on the term structure of interest rates that are implied by only this slight modification of the purely linear case. Moreover, the solution approach that we take should be transferrable to the more general case, but we think that its structure can be made most transparent when concentrating on the special case.

Given the process (2.1) for the short term interest rate, we will now derive bond price processes for all maturities. Let P_t^n denote the time t price of a default-free zero-coupon bond with n periods left until maturity. The payoff is normalized to one, so $P_t^0 = 1$. Continuously compounded monthly yields are computed from bond prices as

$$y_t^n = -\frac{\ln P_t^n}{n}. \tag{2.3}$$

Absence of arbitrage is equivalent to the existence of a strictly positive stochastic discount factor (SDF) process $\{M_t\}$, with $E|M_t P_t^n| < \infty$ and

$$P_t^n = E(M_{t+1} P_{t+1}^{n-1} | \mathcal{F}_t), \tag{2.4}$$

where $\mathcal{F}_t$ denotes the σ-algebra generated by $\{X_{t-i}\}_{i \geq 0}$. Since the short-rate dynamics has the Markov property, any expectation over the future conditional on $\mathcal{F}_t$ equals the expectation conditional on the information contained in X_t alone. We will thus write the basic pricing equation simply as

$$P_t^n = E(M_{t+1} P_{t+1}^{n-1} | X_t). \tag{2.5}$$

For the stochastic discount factor we assume

$$M_{t+1} = \exp\{-\delta - X_t - \lambda\sigma\epsilon_{t+1}\}, \tag{2.6}$$

[6]See Lanne and Saikkonen (2002) for the UK and Switzerland as well as Archontakis and Lemke (2005) for Germany and the US.

where the exponential specification is chosen to guarantee positivity. We set

$$\delta = \frac{1}{2}\sigma^2\lambda^2 \tag{2.7}$$

with hindsight since this specification will lead to y_t^1, the one-month yield, being equal to X_t.[7] The parameter λ is referred to as the market price of risk, it governs the covariance of shocks to the state variable and the discount factor. In affine models, the expected one-period excess return of a long-term bond over the short rate, divided by its standard deviation, is a linear function of λ.[8]

The model specification is now complete: given the state process (2.1) and the pricing kernel specification (2.6), arbitrage free bond price processes $\{P_t^n\}$ are given as the solution of the stochastic difference equation (2.5). An explicit solution of the model writes bond prices, or equivalently yields, as functions of the factor X_t, i.e they are of the form

$$y_t^n = f_n(X_t;\psi), \tag{2.8}$$

where ψ collects all model parameters. The next section is devoted to finding this solution function f_n for our threshold model.

3 Arbitrage-free Term Structure

We start by writing bond prices as a function of future discount factors. Substituting the basic pricing equation (2.5) repeatedly into itself, using the law of iterated expectations and noting that $P_t^0 = 1$, we can write the time t price of the n-period bond as

$$\begin{aligned} P_t^n &= E(M_{t+1}P_{t+1}^{n-1}|X_t) \\ &= E(M_{t+1}E(M_{t+2}P_{t+2}^{n-2}|X_{t+1})|X_t) \\ &= \ldots \\ &= E(M_{t+1}\cdot M_{t+2}\cdot\ldots\cdot M_{t+n}|X_t), \end{aligned} \tag{3.1}$$

equivalently using discount factors in logs,

$$P_t^n = E(\exp[m_{t+1}+\ldots+m_{t+n}]|X_t). \tag{3.2}$$

Before we turn to the model based on (2.1) it is instructive to consider the special case of $\beta = 0$, that is with X_t following the linear Gaussian process

$$X_t = \nu + \kappa X_{t-1} + \sigma\epsilon_t, \quad \epsilon_t \sim N(0,1). \tag{3.3}$$

[7]See Backus et al. (1998).

[8]See Campbell, Lo, and MacKinlay (1997) and Cochrane (2001).

Since for this case X_t is a linear process, the sum of log SDFs can be written as a linear combination of X_t and future ϵ_t only. This yields for the bond price

$$P_t^n = E\left(\exp\left[-a_n - B_n X_t + \sum_{i=1}^{n} b_i^n \epsilon_{t+i}\right] \middle| X_t\right) \tag{3.4}$$

where a_n, B_n and b_n^i are coefficients depending on the model parameters ν, κ, σ, and λ as well as on time to maturity n.[9]

Since the ϵ_t are Gaussian white noise, their sum is also normal and the exponential expression in (3.4) has a conditional lognormal distribution. Computing the required expectation yields the solution[10]

$$P_t^n = \exp[-A_n - B_n X_t], \tag{3.5}$$

where

$$B_n = \sum_{i=0}^{n-1} \kappa^i = \frac{1-\kappa^n}{1-\kappa}, \tag{3.6}$$

$$A_n = \sum_{i=0}^{n-1} G(B_i), \tag{3.7}$$

with

$$G(B_i) = \delta + B_i \nu - \frac{1}{2}(\lambda + B_i)^2 \sigma^2.$$

Using (2.3), we obtain bond yields as an affine function of the short-term interest rate,

$$y_t^n = \frac{A_n}{n} + \frac{B_n}{n} X_t. \tag{3.8}$$

Note that this implies that for a given time to maturity n, the sensitivity of yields with respect to interest rate changes does not depend on the level of the short rate.

We now turn to the case that the short rate follows the threshold process (2.1). That is, the only little difference to the case considered up to now is that the intercept of

[9]The important point is the exponential-affine structure. The exact form of these coefficients, expressed in terms of ν, κ, σ, and λ is not relevant here. For the threshold model (which nests the linear model) they are given in proposition 3.1 below.

[10]Usually, bond prices for the linear Gaussian case are obtained using a method of undetermined coefficients, cf. Backus et al. (1998) or Cochrane (2001). One assumes that bond prices are in fact of the form (3.5) and inserts this expression on both sides of (2.5). It turns out that for A_n and B_n (viewed as a function of n) to satisfy (2.5) for all n and t, they have to solve a system of difference equations the solution of which is given by (3.6) and (3.7). Here we have chosen the approach using the moving average representation in order to parallel it to the solution approach for the threshold case.

the process is allowed to switch endogenously. However, it turns out that this slight modification makes the computation of bond prices a much more intricate task. The following will describe the basic idea of solving for bond prices and state the exact solution. The detailed derivation is delegated to the appendix.

For $n = 1$, we should obtain the short rate itself, i.e. $y_t^1 = X_t$. This is in fact the case since

$$\begin{aligned} P_t^1 &= E(M_{t+1} \cdot 1 | X_t) \\ &= E(\exp[-\delta - X_t - \lambda\sigma\epsilon_{t+1}]) | X_t) \\ &= \exp[-\delta - X_t + \lambda^2\sigma^2], \end{aligned}$$

and thus, using (2.7),

$$y_t^1 = \delta + X_t - \frac{1}{2}\lambda^2\sigma^2 = X_t.$$

For treating maturities $n > 2$ we introduce the notation

$$S_t = I(X_t \geq c) \text{ and } a(S_t) = \nu + \beta S_t,$$

so for the threshold process (2.1),

$$X_{t+1} = a(S_t) + \kappa X_t + \sigma\epsilon_{t+1}. \tag{3.9}$$

The price of the two-period bond is given by

$$\begin{aligned} P_t^2 &= E(\exp[m_{t+1} + m_{t+2}] | X_t) \\ &= E\left(\exp\left[-2\delta - X_t - X_{t+1} - \sigma\lambda(\epsilon_{t+1} + \epsilon_{t+2})\right] | X_t\right) \\ &= \exp\left[-2\delta - X_t - a(S_t) - \kappa X_t\right] \cdot E\left(\exp\left[-\sigma(1+\lambda)\epsilon_{t+1} - \sigma\lambda\epsilon_{t+2}\right] | X_t\right) \end{aligned}$$

Conditional on X_t, the last exponent is normally distributed with mean 0 and variance $\sigma^2((1+\lambda)^2 + \lambda^2)$. Thus, the exponential expression has a conditional lognormal distribution, and

$$E\left(\exp\left[-\sigma(1+\lambda)\epsilon_{t+1} - \sigma\lambda\epsilon_{t+2}\right] | X_t\right) = \exp\left[\frac{1}{2}\sigma^2((1+\lambda)^2 + \lambda^2)\right].$$

Collecting terms delivers

$$P_t^2 = \exp[-A_2(X_t) - B_2 X_t] \tag{3.10}$$

with

$$A_2(X_t) = a(S_t) + 2\delta - \frac{1}{2}\sigma^2(\lambda^2 + (1+\lambda)^2) \tag{3.11}$$

and

$$B_2 = (1+\kappa). \tag{3.12}$$

Hence, using (2.3), for the yield we obtain

$$y_t^2 = \frac{A_2(X_t)}{2} + \frac{B_2}{2} X_t. \tag{3.13}$$

The derivation has employed the same techniques as in the purely Gaussian case. The structure of the solution, however, does differ from (3.8). The two-period yield is a stepwise linear function of the short rate: the intercept depends on $a(S_t) \equiv \nu + \beta \cdot I(X_t \geq c)$. Thus, y_t^2 viewed as a function of X_t features a discontinuity at $X_t = c$. However, at all points of continuity, the derivative of the two-month yield with respect to the short rate is constant. Moreover, the expression B_n is the same as in the linear case.

For $n > 2$ the solution of the bond price can be written in a similar form as in (3.4). However, since the underlying short-rate process now involves the time-varying intercepts, the representation of future log SDFs involves not only future ϵ_t but also future intercepts which in turn are dependent on future X_t. In the appendix it is shown that bond prices can be written as

$$\begin{aligned} P_t^n &= E(\exp[m_{t+1} + \ldots + m_{t+n}]|X_t) \\ &= E\left(\exp\left[-n\delta - B_n X_t + \sum_{i=1}^{n} b_i^n \epsilon_{t+i} + \sum_{j=0}^{n-2} c_j^n \cdot a(S_{t+j}) \right] \middle| X_t \right), \end{aligned}$$

where B_n, b_i^n, c_j^n are coefficients depending on the model parameters κ, σ and λ. The crucial thing to note is that the expression in parentheses is not a linear function of future X_t anymore as it was in the case of a simple linear AR(1) for X_t.[11] Accordingly, conditional on X_t, the expression is not lognormal. Our solution for this case makes use of a similar idea as employed in Bansal and Zhou (2002). We will evaluate the expression by first computing the expectation for an arbitrary *given* realization of $(S_{t+1}, \ldots, S_{t+n-2})'$, say $(\bar{S}_{t+1}, \ldots, \bar{S}_{t+n-2})'$, and then take the probability-weighted sum over all possible realizations of $(S_{t+1}, \ldots, S_{t+n-2})'$. That is, we first enlarge the conditioning information set and then integrate out the enlargement again.

However, even under the extended information set $\{X_t, \bar{S}_{t+1}, \ldots, \bar{S}_{t+n-2}\}$, the exponential does not have a plain lognormal distribution. This is because $(S_{t+1}, \ldots, S_{t+n-2})'$ and $(\epsilon_{t+1}, \ldots, \epsilon_{t+n-2})'$ are not independent. In other words, knowing that a particular path of intercepts $(a(\bar{S}_{t+1}), \ldots, a(\bar{S}_{t+n-2}))'$ has been realized, restricts the set of possible realizations of $(\epsilon_{t+1}, \ldots, \epsilon_{t+n-2})'$: conditional on the extended information set, $(\epsilon_{t+1}, \ldots, \epsilon_{t+n-2})'$ has a *truncated* multivariate lognormal distribution.

[11] Recall that $a(S_t) = \nu + \beta \cdot I(X_t \geq c)$.

A final point to note is that, since S_{t+i} can assume two different values, 1 and 0, the number of different extended information sets, $\{X_t, \bar{S}_{t+1}, \ldots, \bar{S}_{t+n-2}\}$, amounts to 2^{n-2}. It is obvious that this will be one obstacle for obtaining numerical values for bond yields with longer times to maturity.

The following proposition states the solution for bond yields with time to maturity exceeding two months.

Proposition 3.1 (Yield function for $n > 2$). *For the short rate process given by (2.1) and the pricing kernel defined by (2.6), yields with time to maturity $n > 2$ as a function of the short rate X_t are given as:*

$$y_t^n = \frac{A_n(X_t)}{n} + \frac{B_n}{n} X_t \tag{3.14}$$

with

$$B_n = \frac{1-\kappa^n}{1-\kappa} \tag{3.15}$$

and

$$\begin{aligned} & A_n(X_t) \\ = \; & n \cdot \delta - c_0^n a(S_t) - \frac{1}{2} b' b \\ & - \ln \left(\sum_{k=1}^{2^{n-2}} F\left(\tilde{h}(k);\, \tilde{H}(k) b^*,\, \tilde{H}(k)\tilde{H}(k)'\right) \exp \left[\sum_{j=1}^{n-2} c_j^n \cdot a\left(\bar{S}_{t+j}(k)\right) \right] \right) \end{aligned} \tag{3.16}$$

which uses the following definitions:

$$b = (b_1^n, \ldots, b_n^n)', \quad b^* = (b_1^n, \ldots, b_{n-2}^n)' \text{ with } b_i^n = -\sigma \left(\lambda + \frac{1-\kappa^{n-i}}{1-\kappa} \right)$$

$$c_j^n = -\frac{1-\kappa^{n-j-1}}{1-\kappa}, \quad j = 0, 1, \ldots, n-2$$

The function $F(r; \mu, \Sigma)$ denotes the cumulative distribution function of the multivariate normal $N(\mu, \Sigma)$ evaluated at the vector r.

The first summation in (3.16) runs over all possible realizations of the sequence $\{S_{t+1}, \ldots, S_{t+n-2}\}$, i.e. over all possible sequences of length $n-2$ that consist of zeros and ones. $\{\bar{S}_{t+1}(k), \ldots, \bar{S}_{t+n-2}(k)\}$ denotes a particular sequence of this sort. The indexing may be such that k is the decimal number (plus one) that corresponds to the binary number represented by the sequence. For instance, the sequence

$$\{\bar{S}_{t+1}(k), \bar{S}_{t+2}(k), \bar{S}_{t+3}(k), \bar{S}_{t+4}(k)\} = \{1, 0, 0, 1\}$$

corresponds to the decimal number 9 and would carry the index $k = 10(= 9 + 1)$.

The vector $\tilde{h}(k)$ *is given by*[12]

$$\tilde{h}(k) = \tilde{c}(k) - \tilde{f}(k) \cdot X_t - \tilde{G}(k) \cdot a\big(\bar{\zeta}_t^*(k)\big)\,. \tag{3.17}$$

The remaining expressions are defined as follows:

$$\bar{\zeta}_t^*(k) = (S_t, \bar{S}_{t+1}(k), \bar{S}_{t+2}(k), \ldots, \bar{S}_{t+n-3}(k))', \tag{3.18}$$

$$a(\bar{\zeta}_t^*(k)) = (a(S_t), a(\bar{S}_{t+1}(k)), a(\bar{S}_{t+2}(k)), \ldots, a(\bar{S}_{t+n-3}(k)))', \tag{3.19}$$

$$\tilde{f}(k) = \begin{pmatrix} R(\bar{S}_{t+1}(k)) \cdot \kappa^1 \\ \vdots \\ R(\bar{S}_{t+n-2}(k)) \cdot \kappa^{n-2} \end{pmatrix} \tag{3.20}$$

where

$$R(\bar{S}) = \begin{cases} 1, & \textit{if } \bar{S} = 0 \\ -1, & \textit{if } \bar{S} = 1, \end{cases} \tag{3.21}$$

$$\tilde{G}(k) = \begin{pmatrix} \tilde{g}_1^1(k) & 0 & 0 & \ldots & 0 \\ \tilde{g}_1^2(k) & \tilde{g}_2^2(k) & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \tilde{g}_1^{n-2}(k) & \tilde{g}_2^{n-2}(k) & \tilde{g}_3^{n-2}(k) & \ldots & \tilde{g}_{n-2}^{n-2}(k) \end{pmatrix} \tag{3.22}$$

with

$$\tilde{g}_j^i(k) = R(\bar{S}_{t+j}(k)) \cdot \kappa^{i-j}, \tag{3.23}$$

$$\tilde{c}(k) = c \cdot \begin{pmatrix} R(\bar{S}_{t+1}(k)) \\ \vdots \\ R(\bar{S}_{t+n-2}(k)) \end{pmatrix}, \tag{3.24}$$

and

$$\tilde{H}(k) = \begin{pmatrix} \tilde{h}_1^1(k) & 0 & 0 & \ldots & 0 \\ \tilde{h}_1^2(k) & \tilde{h}_2^2(k) & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \tilde{h}_1^{n-2}(k) & \tilde{h}_2^{n-2}(k) & \tilde{h}_3^{n-2}(k) & \ldots & \tilde{h}_{n-2}^{n-2}(k) \end{pmatrix} \tag{3.25}$$

[12]The expressions $\tilde{h}(k)$, $\tilde{c}(k)$, $\tilde{f}(k)$, $\tilde{G}(k)$, $\tilde{c}(k)$, $\tilde{H}(k)$ depend on n and t. However, we omit these arguments in order to avoid an even more messy notation.

with

$$\tilde{h}^i_j(k) = R(\bar{S}_{t+j}(k)) \cdot \sigma\kappa^{i-j}. \tag{3.26}$$

Given the parameters of the threshold model, ν, δ, c, κ, σ and λ, it is in principle possible to compute any n-period yield that corresponds to a realization X_t of the short rate. However, as n gets larger, the following computational obstacles occur.

First, as already mentioned above, the number of different intercept combinations increases exponentially with time to maturity. Second, the formula involves $F\Big(\tilde{h}(k);\, \tilde{H}(k)b^*,\, \tilde{H}(k)\tilde{H}(k)'\Big)$, the c.d.f. of a multivariate normal with general (i.e. non-diagonal) covariance matrix. Numerical software[13] usually has difficulties to compute the corresponding multiple integral for higher dimensions, say exceeding 6. Since computing bond yields of maturity n requires the computation of a c.d.f of an $(n-2)$-variate normal, maturities exceeding eight months cannot be obtained in a straightforward fashion.

In the companion paper Archontakis and Lemke (2005) we circumvent these problems by computing bond prices for higher n employing a simulation-based approach. Conditional on a realization X_t of the short rate we generate N realizations – say N=100,000 – of $(X_{t+1}, \ldots, X_{t+n-1})'$ and $(\epsilon_t, \ldots, \epsilon_{t+n})'$ and compute the corresponding $\exp[m_{t+1} + \ldots m_{t+n}]$. The average of the latter expression over all runs is an estimate of P^n_t, see (3.2). Using the simulation method, that paper explores the properties of bond yields in some detail. In this paper here, we restrict ourselves to illustrate some properties of the yield function for small n which is done in the next section.

4 A Numerical Example

Based on parameter estimates for US data in Archontakis and Lemke (2005), figure 1 draws yields of two-, three- and six-month yields as a function of the one-month rate. The parameters are given as

$$\begin{aligned} \nu = 0.3058/1200, \quad \beta = 0.2603/1200, \quad \kappa = 0.9253, \\ c = 5.5296/1200, \quad \sigma = 0.7136/1200, \quad \lambda = -155. \end{aligned}$$

Recall that for a linear one-factor model, the function that maps the short rate into n-period yields is given by (3.8), i.e. it is affine. For the threshold model, the two-period yield is obtained via (3.13), a stepwise linear function, for $n \geq 3$, the yield function (given

[13]We use GAUSS 6.0 here.

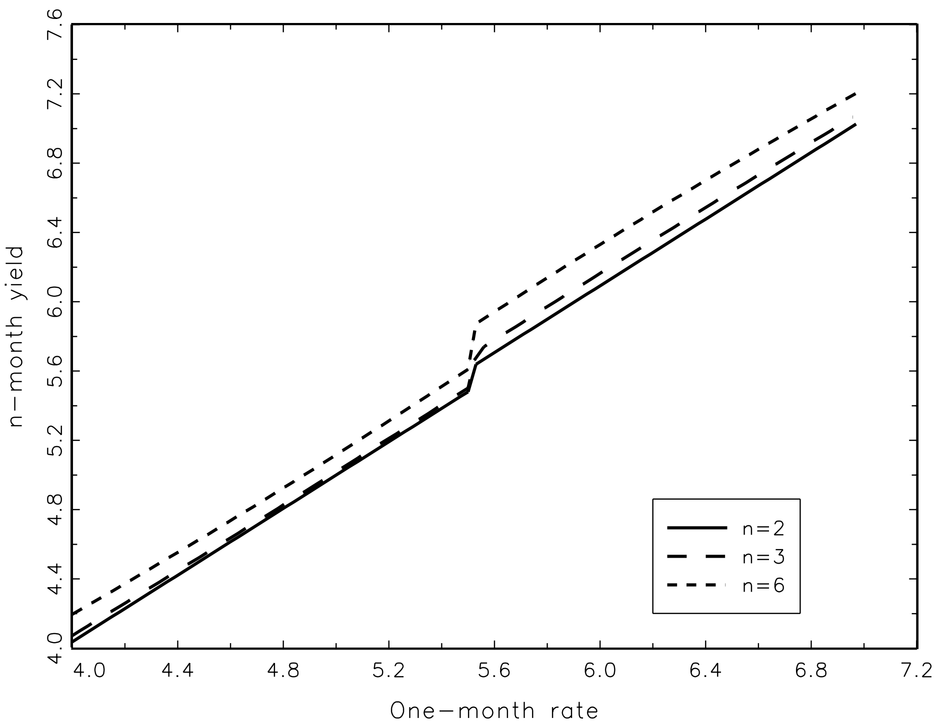

Figure 1: Two-, three- and six-month yield as a function of the short rate.

in proposition 3.1) is nonlinear. It turns out that the 'degree of nonlinearity' increases with time to maturity.[14] However, for small n, for which yields can be actually computed, nonlinearity is hardly visible from the graph.

Therefore, we choose another representation that plots a measure of the second derivative of the yield function against the short rate. Let $f_n(x)$ denote the function that assigns the short rate x the corresponding n-period yield, i.e. $f_n(x) = A_n(x)/n + B_n/n \cdot x$, with $A_n(\cdot)$ and B_n given by (3.16) and (3.15). For a small number h, we approximate the second derivative as

$$\frac{d^2 f_n(x)}{d x^2} \approx \frac{f_n(x-h) - 2f_n(x) + f_n(x+h)}{h^2} =: k_n(x), \tag{4.1}$$

at all points of continuity. That is, we do not compute $k_n(x)$ if $[x-h, x+h]$ contains the threshold value c. Figure 2 plots $k_n(x)$ against x for $n = 2, 3$ and 6.

For $n = 2$ the function is identically zero since $f_2(x)$ is stepwise linear, so the second derivative disappears at all points of continuity. For $n = 3$ and $n = 6$ the figure shows that

[14] See Archontakis and Lemke (2005).

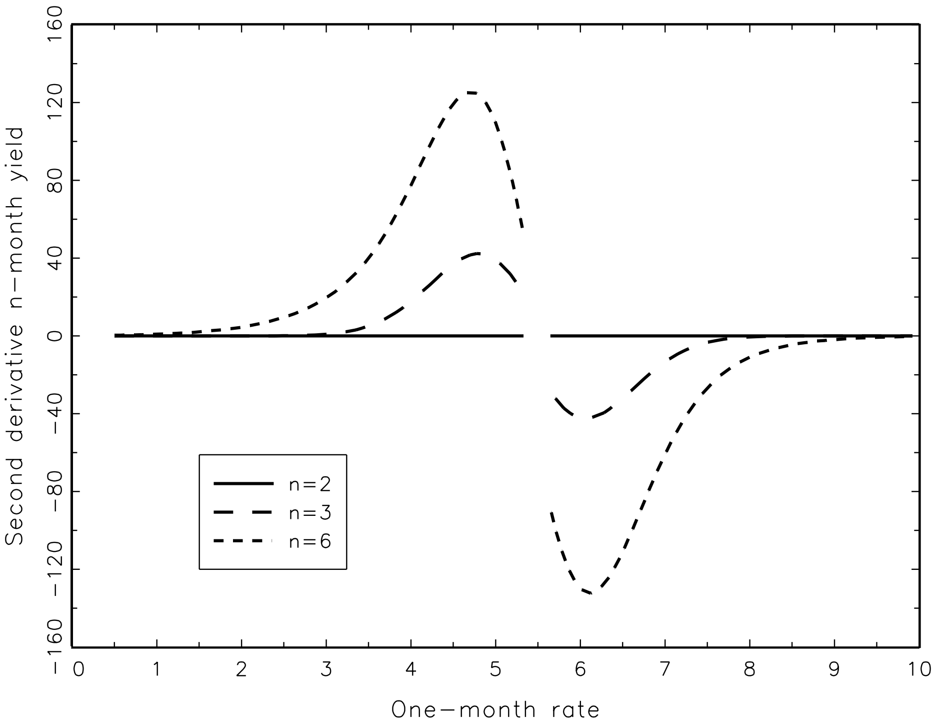

Figure 2: Second derivative of the yield function against the short rate.

there is in fact a nonlinearity around the threshold value (that could not be made visible in figure 1). In particular, the yield functions f_3 and f_6 exhibit a convex-concave pattern: on the left of the threshold value the sensitivity of y^n with respect to x increases (positive second derivative, i.e. increase in (positive) first derivative), on the right it decreases (negative second derivative, i.e. decrease in (positive) first derivative). Moreover, the interval in which 'nonlinearity prevails' is bigger for $n = 6$ than for $n = 3$. As shown in Archontakis and Lemke (2005) it tends to rise monotonically with time to maturity.

5 Summary and Outlook

Assuming a linear Gaussian process for the short-term interest rate and an adequately chosen pricing kernel induces bond yields to be affine functions of the short rate under the condition of no-arbitrage. While this is a convenient property of linear models, the empirical literature on interest rate dynamics finds evidence for nonlinearities in short-rate dynamics. This poses the question how certain forms of nonlinear dynamics translate into the cross-sectional relationship between bond yields of different maturities. This paper considered a very simple case of a nonlinear specification for the one-month rate: a SETAR process that allows the intercept parameter to switch, while all other parameters are assumed to be constant. This specification has been proposed by Lanne and Saikkonen (2002) as it is especially suited to capture the near-unit-root behavior of interest rates.

We derive the exact formula for bond yields as a function of the one-month rate. It turns out that the yield functions exhibit a convex-concave pattern around the threshold value. The region in which nonlinearity prevails increases with time to maturity. That is, for small time to maturity n, the corresponding yield as a function of the short rate is nearly (stepwise) linear.

While our solution for bond prices delivers the exact solution in a finite number of operations, two numerical problems arise for the computation of yields for longer (i.e. $n > 7$ months) time to maturity. First, the solution for the n-period yield requires the computation of the c.d.f. of an $(n-2)$-variate normal with general variance-covariance matrix, which is usually not feasible to be computed. The second problem lies in the fact that the computational burden increases exponentially with time to maturity. The companion paper Archontakis and Lemke (2005) circumvents these problems by relying on a pure simulation-based approach for longer times to maturity. For future research it is conceivable to use more clever numerical routines to compute the required c.d.f.s. Concerning the curse of dimensionality problem, one may employ a mixed approach that makes use of our analytical solution within a simulation-based approach.

The solution approach introduced in this paper can be transferred to richer nonlinear models of the term structure. For instance, parameters other than intercepts may be allowed to switch as well. This includes models for which the degree of mean reversion depends on the level of interest rates. Moreover, the technique introduced in this paper should be easily transferable to multifactor models. If solution functions for those models can actually be made computable, those nonlinear models should be compared to the well-established multifactor affine models.

A Two Auxiliary Results

We first provide two auxiliary results about the expectations of the (truncated) multivariate log-normal distribution. Let x be distributed as an m-variate normal, $x \sim N(0, \Omega)$, and let d and r be vectors of length m. Then:

$$E(\exp[d'x]) = \exp\left[\frac{1}{2}d'\Omega d\right] \tag{A.1}$$

and

$$E(\exp[d'x]|x < r) = \frac{1}{Pr(x < r)} F(r; \Omega d, \Omega) \exp\left[\frac{1}{2}d'\Omega d\right], \tag{A.2}$$

where $F(r; \Omega\, d, \Omega)$ denotes the c.d.f. of the multivariate normal with mean Ωd and variance-covariance matrix Ω evaluated at r.

Noting that $d'x$ is a scalar normal random variable, the first expression is a standard result. To show the second result, first note that the conditional density required to compute the expectation is given by

$$p(x|x < r) = \frac{p(x)}{Pr(x < r)}.$$

where $p(x)$ is the density of the normal $N(0, \Omega)$ and $Pr(x < r)$ is the c.d.f. of that normal evaluated at r.[15]

Then we have

$$\begin{aligned}
& E(\exp[d'x]|x < r) \\
= \; & \frac{1}{Pr(x < r)} \int_{-\infty}^{r_m} \cdots \int_{-\infty}^{r_1} \frac{1}{(2\pi)^{(m/2)}} |\Omega|^{-1/2} \exp[-1/2x'\Omega^{-1}x] \; \exp[d'x] \, dx_1 \ldots dx_m \\
= \; & \frac{1}{Pr(x < r)} \int_{-\infty}^{r_m} \cdots \int_{-\infty}^{r_1} \frac{1}{(2\pi)^{(m/2)}} |\Omega|^{-1/2} \\
& \times \exp[(-1/2)(x'\Omega^{-1}x - 2d'x - d'\Omega d + d'\Omega d)] \, dx_1 \ldots dx_m \\
= \; & \frac{1}{Pr(x < r)} \int_{-\infty}^{r_m} \cdots \int_{-\infty}^{r_1} \frac{1}{(2\pi)^{(m/2)}} |\Omega|^{-1/2} \\
& \times \exp[(-1/2)(x - \Omega d)'\Omega^{-1}(x - \Omega d) + (1/2)d'\Omega d] \, dx_1 \ldots dx_m \\
= \; & \frac{1}{Pr(x < r)} \cdot \exp[(1/2)d'\Omega d] \cdot \int_{-\infty}^{r_m} \cdots \int_{-\infty}^{r_1} \frac{1}{(2\pi)^{(m/2)}} |\Omega|^{-1/2} \\
& \times \exp[(-1/2)(x - \Omega d)'\Omega^{-1}(x - \Omega d)] \, dx_1 \ldots dx_m \\
= \; & \frac{1}{Pr(x < r)} \cdot \exp[(1/2)d'\Omega d] \cdot F(r; \Omega d, \Omega).
\end{aligned}$$

[15] So we could write here and in (A.2) $F(r; 0, \Omega)$ instead of $Pr(x < r)$. However, we stick to $Pr(x < r)$ since this is a more convenient notation for the derivation following in section B.

B Derivation of the Bond Pricing Formula for $n > 2$

1. Representation of X_{t+i} and partial sums of log SDFs

In the following we will need X_{t+i} written in terms of X_t, future ϵ_t and future intercepts as well as partial sums of the pricing kernel M_t.

Starting with X_t and iterating (3.9) forward leads to

$$\begin{aligned} X_{t+i} &= \kappa^i X_t + \kappa^{i-1} a(S_t) + \kappa^{i-2} a(S_{t+1}) + \ldots + \kappa a(S_{t+i-2}) + a(S_{t+i-1}) \\ &\quad + \sigma\kappa^{i-1}\epsilon_{t+1} + \sigma\kappa^{i-2}\epsilon_{t+2} + \ldots + \sigma\kappa\epsilon_{t+i-1} + \sigma\epsilon_{t+i}, \end{aligned}$$

in compact form

$$X_{t+i} = \kappa^i X_t + \sum_{l=1}^{i} g_l^i a(S_{t+l-1}) + h_l^i \epsilon_{t+l}. \tag{B.3}$$

For partial sums of X_t we obtain

$$\begin{aligned} & X_{t+1} + X_{t+2} + \ldots + X_{t+m} \\ = \; & (\kappa + \kappa^2 + \ldots + \kappa^m) X_t \\ & + (1 + \kappa + \ldots + \kappa^{m-1}) a(S_t) + (1 + \kappa + \ldots + \kappa^{m-2}) a(S_{t+1}) + \ldots \\ & + (1 + \kappa) a(S_{t+m-2}) + a(S_{t+m-1}) \\ & + \sigma(1 + \kappa + \ldots + \kappa^{m-1})\epsilon_{t+1} + \sigma(1 + \kappa + \ldots + \kappa^{m-2})\epsilon_{t+2} + \ldots \\ & + \sigma(1+\kappa)\epsilon_{t+m-1} + \sigma\epsilon_{t+m}, \end{aligned}$$

Using the latter result, the sum of the log discount factors, $m_t = \ln(M_t)$, can be written as

$$\begin{aligned} & m_{t+1} + m_{t+2} + \ldots + m_{t+n} \\ = \; & -n\delta - X_t - X_{t+1} - \ldots - X_{t+n-1} - \sigma\lambda\epsilon_{t+1} - \sigma\lambda\epsilon_{t+2} - \ldots - \sigma\lambda\epsilon_{t+n} \\ = \; & -n\delta - (1 + \kappa + \ldots + \kappa^{n-1}) X_t \\ & - (1 + \kappa + \ldots + \kappa^{n-2}) a(S_t) - (1 + \kappa + \ldots + \kappa^{n-3}) a(S_{t+1}) - \ldots \\ & - (1+\kappa) a(S_{t+n-3}) - a(S_{t+n-2}) \\ & - \sigma(\lambda + 1 + \kappa + \ldots + \kappa^{n-2})\epsilon_{t+1} - \sigma(\lambda + 1 + \kappa + \ldots + \kappa^{n-2})\epsilon_{t+2} - \ldots \\ & - \sigma(\lambda + 1)\epsilon_{t+n-1} - \sigma\lambda\epsilon_{t+n-2}, \end{aligned}$$

compactly,

$$m_{t+1} + \ldots + m_{t+n} = -n\delta - B_n X_t + \sum_{i=1}^{n} b_i^n \epsilon_{t+i} + \sum_{j=0}^{n-2} c_j^n \cdot a(S_{t+j}). \tag{B.4}$$

2. Bond price as product of three factors

We plug (B.4) into the bond price formula (3.2) and obtain

$$\begin{aligned} P_t^n &= E(\exp[m_{t+1} + \ldots + m_{t+n}]|X_t) \\ &= E\left(\exp\left[-n\delta - B_n X_t + \sum_{i=1}^{n} b_i^n \epsilon_{t+i} + \sum_{j=0}^{n-2} c_j^n \cdot a(S_{t+j})\right]\middle| X_t\right). \end{aligned}$$

The random variables X_t and S_t are part of the conditioning information set and can thus be taken outside the expectation. (Note that knowing X_t implies knowing if $X_t < c$ is true and thus knowing the realization of $S_t = I(X_t \geq c)$.) Moreover, ϵ_{t+n-1} and ϵ_{t+n} are independent of $(S_t, S_{t+1}, S_{t+n-2}, \epsilon_{t+1}, \ldots, \epsilon_{t+n-2})'$. Hence, we can write

$$\begin{aligned} P_t^n &= \exp[-n\delta - B_n X_t + c_0^n a(S_t)] \\ &\times\ E(\exp[b_{n-1}^n \epsilon_{t+n-1} + b_n^n \epsilon_{t+n}]|X_t) \\ &\times\ E\left(\exp\left[\sum_{i=1}^{n-2} b_i^n \epsilon_{t+i} + c_i^n a(S_{t+i})\right]\middle| X_t\right). \end{aligned} \tag{B.5}$$

The product consists of three factors. The first factor contains only quantities known at time t. The expectation of the second factor can be computed using the first of our auxiliary results, (A.1), since $(\epsilon_{t+n-1}, \epsilon_{t+n})'$ is conditionally (and unconditionally) normally distributed. Thus, using the terms of (A.1) we have $d = (b_{n-1}^n, b_n^n)'$, $x = (\epsilon_{t+n-1}, \epsilon_{t+n})'$, $\mu = 0_2$, and $\Omega = I_2$ and we obtain for the second factor in (B.5)

$$E(\exp[b_{n-1}^n \epsilon_{t+n-1} + b_n^n \epsilon_{t+n}]|X_t) = \exp[0.5(b_{n-1}^n)^2 + 0.5(b_n^n)^2]. \tag{B.6}$$

3. Computation of $E\left(\exp\left[\sum_{i=1}^{n-2} b_i^n \epsilon_{t+i} + \sum_{j=1}^{n-2} c_j^n a(S_{t+j})\right]\middle| X_t\right)$.

For computing the third factor in (B.5) it is important to note that $(S_{t+1}, \ldots, S_{t+n-2})'$ and $(\epsilon_{t+1}, \ldots, \epsilon_{t+n-2})'$ are not independent. We will evaluate the expression by first computing the expectation for an arbitrary *given* realization of $(S_{t+1}, \ldots, S_{t+n-2})'$ and then take the probability-weighted sum over all possible realizations of $(S_{t+1}, \ldots, S_{t+n-2})'$. That is, we first enlarge the conditioning information set and then integrate out the enlargement again.[16]

Let

$$\bar{\zeta}_t = (\bar{S}_{t+1}, \ldots, \bar{S}_{t+n-2})'$$

[16] A similar approach is taken by Bansal and Zhou (2002), deriving bond prices for the case that the state evolution is subject to Markov regime switching.

denote a realization of

$$\zeta_t = (S_{t+1}, \ldots, S_{t+n-2})',$$

i.e $\bar{\zeta}_t$ is a sequence of zeros and ones. There are 2^{n-2} different such sequences. They will be indexed $k = 1, 2, \ldots, 2^{n-2}$ such that $k-1$ is that decimal number that corresponds to the binary number represented by $\bar{\zeta}_t$. For example for $n = 6$, $\bar{\zeta}_t(k = 1) = (0, 0, 0, 0)'$ and $\bar{\zeta}_t(k = 14) = (1, 1, 0, 1)'$.

Thus, we have

$$\begin{aligned} & E\left(\exp\left[\sum_{i=1}^{n-2} b_i^n \epsilon_{t+i} + c_i^n a(S_{t+i}) \right] \middle| X_t \right) \\ = & \sum_{k=1}^{2^{n-2}} Pr\big(\bar{\zeta}_t(k) \big| X_t\big) \; E\left(\exp\left[\sum_{i=1}^{n-2} b_i^n \epsilon_{t+i} + c_i^n a(\bar{S}_{t+i}(k)) \right] \middle| X_t, \bar{\zeta}_t(k) \right) \end{aligned}$$

where $Pr(\bar{\zeta}_t(k)|X_t)$ denotes the conditional probability of the realization $\zeta_t = \bar{\zeta}_t(k)$.

For the expectation conditional on the augmented information set we can pull out expressions involving $\bar{S}_{t+i}$, hence

$$\begin{aligned} & E\left(\exp\left[\sum_{i=1}^{n-2} b_i^n \epsilon_{t+i} + c_i^n a(S_{t+i}) \right] \middle| X_t \right) \\ = & \sum_{k=1}^{2^{n-2}} Pr\big(\bar{\zeta}_t(k) \big| X_t\big) \exp\left[\sum_{i=1}^{n-2} c_i^n a(\bar{S}_{t+i}(k)) \right] \\ & \times E\left(\exp\left[\sum_{i=1}^{n-2} b_i^n \epsilon_{t+i} \right] \middle| X_t, \bar{\zeta}_t(k) \right). \end{aligned} \tag{B.7}$$

4. Computation of $E\left(\exp\left[\sum_{i=1}^{n-2} b_i^n \epsilon_{t+i}\right] \big| X_t, \bar{\zeta}_t(k)\right)$.

In order to compute the last conditional expectation appearing in the latter expression we will make use of our auxiliary result (A.2). For this we will rewrite the conditioning information set as a set of inequality conditions.

To explain the approach, we consider the following example. If, for $n = 5$, $\bar{\zeta}_t(3) = (0, 1, 0)$, this is equivalent to the event

$$X_{t+1} < c, X_{t+2} \geq c, X_{t+3} < c.$$

Making use of (B.3), these three inequalities can be written as

$$\begin{aligned} \kappa X_t + g_1^1 a(S_t) + h_1^1 \epsilon_{t+1} &< c \\ \kappa^2 X_t + g_1^2 a(S_t) + a_2^2 a(\bar{S}_{t+1}(k)) + h_1^2 \epsilon_{t+1} + h_2^2 \epsilon_{t+2} &\geq c \\ \kappa^3 X_t + g_1^3 a(S_t) + g_2^3 a(\bar{S}_{t+1}(k)) + g_3^3 a(\bar{S}_{t+2}(k)) + h_1^3 \epsilon_{t+1} + h_2^3 \epsilon_{t+2} + h_3^3 \epsilon_{t+3} &< c \end{aligned}$$

To be able to apply our auxiliary result (A.2) we only want to have '$<$' inequalities. So we multiply every '$\geq$' inequality by -1. Technically, we multiply through any inequality by a factor $R(\bar{S}_{t+i}(k))$, where for the function $R(\cdot)$ defined on $\{0,1\}$, $R(0) = 1$, and $R(1) = -1$. Hence, in the above example $R(\bar{S}_{t+1}(k)) = R(0) = 1$, $R(\bar{S}_{t+2}(k)) = -1$, and $R(\bar{S}_{t+3}(k)) = 1$. Thus, the inequality corresponding to a particular $\bar{S}_{t+i}(k)$ is written as

$$R(\bar{S}_{t+i}(k)) \cdot \left[\kappa^i X_t + \sum_{l=1}^{i} g_l^i a(\bar{S}_{t+l-1}) + h_l^i \epsilon_{t+l}\right] < R(\bar{S}_{t+i}(k))c. \tag{B.8}$$

Accordingly, the set of inequalities corresponding to a particular $(\bar{S}_{t+1}(k), \ldots, \bar{S}_{t+n-2}(k))'$ can be written in vector-matrix notation as

$$\begin{aligned}
& \begin{pmatrix} R(\bar{S}_{t+1}(k)) \\ R(\bar{S}_{t+2}(k)) \\ \vdots \\ R(\bar{S}_{t+n-2}(k)) \end{pmatrix} \odot \left(\begin{pmatrix} \kappa \\ \kappa^2 \\ \vdots \\ \kappa^{n-2} \end{pmatrix} X_t \right. \\
+ & \begin{pmatrix} g_1^1 & 0 & 0 & \ldots & 0 \\ g_1^2 & g_2^2 & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ g_1^{n-2} & g_2^{n-2} & g_3^{n-2} & \ldots & g_{n-2}^{n-2} \end{pmatrix} \begin{pmatrix} a(S_t) \\ a(\bar{S}_{t+1}(k)) \\ \vdots \\ a(\bar{S}_{t+n-3}(k)) \end{pmatrix} \\
+ & \left. \begin{pmatrix} h_1^1 & 0 & 0 & \ldots & 0 \\ h_1^2 & h_2^2 & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ h_1^{n-2} & h_2^{n-2} & h_3^{n-2} & \ldots & h_{n-2}^{n-2} \end{pmatrix} \begin{pmatrix} \epsilon_{t+1} \\ \epsilon_{t+2} \\ \vdots \\ \epsilon_{t+n-2} \end{pmatrix} \right) \\
< \;\; & c \cdot \begin{pmatrix} R(\bar{S}_{t+1}(k)) \\ R(\bar{S}_{t+2}(k)) \\ \vdots \\ R(\bar{S}_{t+n-2}(k)) \end{pmatrix}
\end{aligned}$$

where '$\odot$' denotes elementwise multiplication of two vectors. Using the definitions (3.17) - (3.26), and $\mathcal{E}_t = (\epsilon_{t+1}, \ldots, \epsilon_{t+n-2})'$ this is written compactly as

$$\tilde{f}(k)X_t + \tilde{G}(k)a(\bar{\zeta}_t^*(k)) + \tilde{H}(k)\mathcal{E}_t < \tilde{c}(k) \tag{B.9}$$

or

$$\tilde{H}(k)\mathcal{E}_t < \tilde{h}(k). \tag{B.10}$$

It is important to note that multiplying both sides of (B.10) by the inverse of $\tilde{H}(k)$ would not be an equivalent transformation of that inequality.[17] We define a new random vector

$$\tilde{z}(k) = \tilde{H}(k)\,\mathcal{E}_t.$$

Since $\mathcal{E}_t \sim N(0_{n-2}, I_{n-2})$, we have

$$\tilde{z}(k) \sim N(0, \tilde{H}(k)\tilde{H}(k)')$$

Now we can turn the expression to be computed,

$$E\left(\exp\left[b^{*\prime}\mathcal{E}_t\right]\middle|\, X_t, \bar{\zeta}_t(k)\right),$$

into the form of (A.2).[18] That is we rewrite the exponential in terms of $\tilde{z}(k)$ and the conditioning on $\bar{\zeta}_t(k)$ in terms of an inequality for $\tilde{z}(k)$. Then we apply (A.2). We obtain

$$\begin{aligned}
& E\left(\exp\left[b^{*\prime}\mathcal{E}_t\right]\middle|\, X_t, \bar{\zeta}_t(k)\right) \\
= \;& E\left(\exp\left[(\tilde{H}(k)^{-1\prime}b^*)'\tilde{z}(k)\right]\middle|\, X_t,\ \tilde{z}(k) < \tilde{h}(k)\right) \\
= \;& \frac{1}{Pr(\tilde{z}(k) < \tilde{h}(k)|X_t)} \\
& \times \exp\left[0.5\, b^{*\prime}\tilde{H}(k)^{-1}\tilde{H}(k)\tilde{H}(k)'\tilde{H}(k)^{-1\prime}b^*\right] \\
& \times F\left(\tilde{h}(k); \tilde{H}(k)\tilde{H}(k)'\tilde{H}(k)^{-1\prime}b^*,\ \tilde{H}(k)\tilde{H}(k)'\right) \\
= \;& \frac{1}{Pr(\tilde{z}(k) < \tilde{h}(k)|X_t)} \\
& \times \exp\left[0.5\, b^{*\prime}b^*\right] \\
& \times F\left(\tilde{h}(k); \tilde{H}(k)b^*,\ \tilde{H}(k)\tilde{H}(k)'\right)
\end{aligned}$$

Finally note that

$$Pr(\tilde{z}(k) < \tilde{h}(k)|X_t) = Pr(\bar{\zeta}_t(k)|X_t),$$

since $\{\tilde{z}(k) < \tilde{h}(k)|X_t\}$ and $\{\bar{\zeta}_t(k)|X_t\}$ are equivalent events as we derived above.

5. Putting things together

[17] As a simple example, one can easily verify that $Ax < c$ – with $A = \begin{pmatrix} a_1 & 0 \\ a_2 & a_3 \end{pmatrix}$, $x = (x_1, x_2)'$, $c = (c_1, c_2)'$, a_1, a_2, a_3, c_1, c_2 all positive – defines a different region in x_1, x_2 space than $x < A^{-1}c$.

[18] Note that the only slight difference to (A.2) is that everything is conditional on X_t. However, a 'conditional version' of (A.2) could be derived in the same way as the unconditional version.

In step 4 we computed the last term in (B.7). Plugging in we obtain

$$
\begin{aligned}
& E\left(\exp\left[\sum_{i=1}^{n-2} b_i^n \epsilon_{t+i} + c_i^n a(S_{t+i})\right]\middle| X_t\right) \\
= & \sum_{k=1}^{2^{n-2}} \exp\left[\sum_{j=1}^{n-2} c_j^n a(\bar{S}_{t+j}(k))\right] Pr\big(\bar{\zeta}_t(k)\big| X_t\big) \\
& \times E\left(\exp\left[\sum_{i=1}^{n-2} b_i^n \epsilon_{t+i}\right]\middle| X_t, \bar{\zeta}_t(k)\right) \\
= & \sum_{k=1}^{2^{n-2}} \exp\left[\sum_{j=1}^{n-2} c_j^n a(\bar{S}_{t+j}(k))\right] Pr\big(\bar{\zeta}_t(k)\big| X_t\big) \\
& \times \frac{1}{Pr\big(\bar{\zeta}_t(k)\big| X_t\big)} \cdot \exp\left[0.5\, b^{*\prime} b^*\right] \cdot F\left(\tilde{h}(k); \tilde{H}(k) b^*,\ \tilde{H}(k)\tilde{H}(k)'\right) \\
= & \sum_{k=1}^{2^{n-2}} \exp\left[\sum_{j=1}^{n-2} c_j^n a(\bar{S}_{t+j}(k))\right] \\
& \times \exp\left[0.5\, b^{*\prime} b^*\right] \cdot F\left(\tilde{h}(k); \tilde{H}(k) b^*,\ \tilde{H}(k)\tilde{H}(k)'\right)
\end{aligned}
$$

Using the latter and (B.6) we obtain for the bond price (B.5),

$$
\begin{aligned}
P_t^n \;=\; & \exp[-n\delta - B_n + X_t + c_0^n a(S_t)] \\
& \times E(\exp[b_{n-1}^n \epsilon_{t+n-1} + b_n^n \epsilon_{t+n}] | X_t) \\
& \times E\left(\exp\left[\sum_{i=1}^{n-2} b_i^n \epsilon_{t+i} + c_i^n a(S_{t+i})\right]\middle| X_t\right) \\
= \; & \exp[-n\delta - B_n X_t + c_0^n a(S_t)] \cdot \exp[0.5 (b_{n-1}^n)^2 + 0.5 (b_n^n)^2] \\
& \times \sum_{k=1}^{2^{n-2}} \exp\left[\sum_{j=1}^{n-2} c_j^n a(\bar{S}_{t+j}(k))\right] \\
& \times \exp\left[0.5\, b^{*\prime} b^*\right] \cdot F\left(\tilde{h}(k); \tilde{H}(k) b^*,\ \tilde{H}(k)\tilde{H}(k)'\right) \\
= \; & \exp[-n\delta - B_n X_t + c_0^n a(S_t)] \cdot \exp[0.5\, b' b] \\
& \times \sum_{k=1}^{2^{n-2}} \exp\left[\sum_{j=1}^{n-2} c_j^n a(\bar{S}_{t+j}(k))\right] F\left(\tilde{h}(k); \tilde{H}(k) b^*,\ \tilde{H}(k)\tilde{H}(k)'\right)
\end{aligned}
$$

Transferring the price into a yield using (2.3) completes the proof.

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