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Recursive robust estimation and control without commitment

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Abstract

In a Markov decision problem with hidden state variables, a posterior distribution serves as a state variable and Bayes' law under an approximating model gives its law of motion. A decision maker expresses fear that his model is misspecified by surrounding it with a set of alternatives that are nearby when measured by their expected log likelihood ratios (entropies). Martingales represent alternative models. A decision maker constructs a sequence of robust decision rules by pretending that a sequence of minimizing players choose increments to a martingale and distortions to the prior over the hidden state. A risk sensitivity operator induces robustness to perturbations of the approximating model conditioned on the hidden state. Another risk sensitivity operator induces robustness to the prior distribution over the hidden state. We use these operators to extend the approach of Hansen and Sargent (1995) to problems that contain hidden states. The worst case martingale is overdetermined, expressing an intertemporal inconsistency of worst case beliefs about the hidden state, but not about observables.

Non-technical summary

This paper deals with a fundamental question of applied economics: how should decision-makers, especially economic policy decision-makers, behave if they wish to take account of the fact that their knowledge of the economy is no more than incomplete. This is of key importance for central banks, which continually have to take monetary policy decisions that are necessarily based on models, ie on systematic simplifications of reality, the precise details of which can never be understood with complete certainty.

This problem can be specifically related to the debate that has been conducted on the risk of deflation in the USA and Europe. In 2003, short-term nominal interest rates were at an all-time low, as was inflation. In this context, it should be remembered that nominal interest rates cannot be negative and that the real interest rates which are relevant to economic planning are nominal interest rates less inflation. Statistical models which have been used up to now, and which have been used by central banks very successfully for forecasts under conditions of strictly positive inflation, may turn out to be unsuitable for new, more extreme conditions. For example, given deflation (in other words, negative inflation), if nominal interest rates are almost equal or equal to zero, real interest rates are, of necessity, positive. These, in turn, slow down the economy and may therefore further accelerate deflation. A traditional model might give disastrous recommendations even though it has functioned well in "normal" times.

This is one of the reasons why central banks do not rely blindly on statistical models, but rather draw on their experience and intuition under rarer but riskier conditions. It may rightly be claimed that deflation in the USA has been prevented *inter alia* by the Fed acting in a forward-looking manner and attaching particular importance to worst case scenarios, even though traditional models make no provision for these.

What position does this paper take up in this context? It develops principles on which economic policymakers can draw if the "true" model of the world is unknown to them. It thereby departs from the existing analytical methodology which studies the economic policy issues on the assumption that all the economic agents know the "correct" model and that everyone knows the same, correct model. Although conducting analyses on this assumption has enormous practical advantages, there is no perception of how risky the unconscious use of a possibly incorrect model is under extreme conditions, even if such conditions are very unlikely.

Nicht technische Zusammenfassung

Diese Arbeit beschäftigt sich mit einer fundamentalen Frage der angewandten Volkswirtschaftslehre: Wie sollen sich Entscheidungsträger, insbesondere in der Wirtschaftspolitik, verhalten, wenn sie berücksichtigen wollen, dass sie die Wirtschaft nur unvollständig kennen. Dies ist von zentraler Bedeutung für Zentralbanken, die kontinuierlich geldpolitische Entscheidungen treffen müssen, die notwendigerweise auf Modellen beruhen, also auf systematischen Vereinfachungen der Wirklichkeit, deren Zusammenhänge man notwendigerweise nie mit völliger Sicherheit wird verstehen können.

Konkret lässt sich dieses Problem auf die Diskussion anwenden, die in Bezug auf die Gefahr einer Deflation in den USA und Europa geführt wurde. Kurzfristige Nominalzinsen waren 2003 auf historisch niedrigem Niveau, ebenso die Inflation. Man bedenke dabei, dass Nominalzinsen nicht negativ sein können, und dass die für wirtschaftliche Planungen relevanten Realzinsen gleich Nominalzins minus Inflation sind. Bisher benutzte statistische Modelle, die von Zentralbanken unter Bedingungen strikt positiver Inflation sehr gut für Vorhersagen verwendet werden konnten, sind möglicherweise für neue, extremere, Bedingungen nicht geeignet. Wenn zum Beispiel bei einer Deflation (also negativer Inflation) die nominellen Zinsen fast oder gleich Null sind, liegen notwendigerweise positive Realzinsen vor. Diese wiederum bremsen die Konjunktur und können somit eine Deflation noch beschleunigen. Ein herkömmliches Modell könnte unter Umständen katastrophale Empfehlungen geben, obwohl es in „normalen“ Zeiten gut funktioniert hat.

Dies ist einer der Gründe, warum Zentralbanken nicht blind auf statistische Modelle vertrauen, sondern unter seltenen, aber risikoreichen Bedingungen auf ihre Erfahrung und Intuition zurückgreifen. Es kann mit Recht behauptet werden, dass eine Deflation in den USA auch dadurch vermieden wurde, dass die Fed vorausschauend agiert hat, und „worst case“ Szenarien besonderes Gewicht verliehen hat, obwohl herkömmliche Modelle diese nicht vorsehen.

Wie ordnet sich diese Arbeit in diese Diskussion ein? Sie erarbeitet Prinzipien, auf die wirtschaftliche Entscheidungsträger zurückgreifen können, wenn ihnen das „wahre“ Modell der Welt nicht bekannt ist. Damit weicht sie von der bisherigen Analysemethodik ab, die wirtschaftspolitische Fragestellungen unter der Annahme untersucht, dass allen Wirtschaftssubjekten das „richtige“ Modell bekannt ist, und dass alle das gleiche, richtige Modell kennen. Während es enorme praktische Vorteile hat, Analysen unter dieser Annahme vorzunehmen, hat man kein Gefühl dafür, wie riskant der unbewusste Gebrauch eines vielleicht falschen Modells unter extremen Bedingungen ist, selbst wenn diese sehr unwahrscheinlich sind.

Recursive Robust Estimation and Control Without Commitment *

1 Introduction

In problems with incomplete information, optimal decision rules depend on a decision maker's posterior distribution over hidden state variables, called $q_t(z)$ here, an object that summarizes the history of observed signals. A decision maker expresses faith in his model when he uses Bayes' rule to deduce the transition law for $q_t(z)$.[1]

But how should a decision maker proceed if he doubts his model and wants a decision rule that is robust to a set of statistically difficult to detect misspecifications of it? We begin by assuming that, through some unspecified process, a decision maker has arrived at an

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[1]For example, see Jovanovic (1979), Jovanovic and Nyarko (1995), Jovanovic and Nyarko (1996), and Bergemann and Valimaki (1996).

approximating model that fits historical data well. Because he fears that his approximating model is misspecified, he surrounds it with a set of all alternative models whose expected log likelihood ratios (i.e., whose *relative entropies*) are restricted or penalized. The decision maker believes that the data will be generated by an unknown member of this set. When relative entropies are constrained to be small, the decision maker believes that his model is a good approximation. The decision maker wants robustness against these alternatives because, as Anderson, Hansen, and Sargent (2003) emphasize, perturbations with small relative entropies are statistically difficult to distinguish from the approximating model. This paper assumes that the appropriate summary of signals continues to be the decision maker's posterior under the approximating model, despite the fact that he distrusts that model. Hansen and Sargent (2005) explore the meaning of this assumption by studying a closely related decision problem under commitment to a worst case model.

Section 2 formulates a Markov control problem in which a decision maker with a trusted model receives signals about hidden state variables. By allowing the hidden state vector to index submodels, this setting includes situations in which the decision maker has multiple models or is uncertain about coefficients in those models. Subsequent sections view the model of section 2 as an approximation, use relative entropy to define a cloud of models that are difficult to distinguish from it statistically, and construct a sequence of decision rules that can work well for all of those models. Section 3 uses results of Hansen and Sargent (2005) to represent distortions of an approximating model in terms of martingales defined on the same probability space as the approximating model. Section 4 then defines two operators, T_1 and T_2, respectively, that are indexed by penalty parameters (θ_1, θ_2). In section 5, we use T^1 to adjust continuation values for concerns about model misspecification, conditioned on knowledge of the hidden state. We use T^2 to adjust continuation values for concern about misspecification of the distribution of the hidden state. We interpret θ_1 and θ_2 as penalties on pertinent entropy terms

Section 6 discusses the special case that prevails when $\theta_1 = \theta_2$ and relates it to a decision problem under commitment that we analyzed in Hansen and Sargent (2005). We discuss the dynamic consistency of worst case beliefs about the hidden state in subsections 6.4 and 6.6. Section 7 describes the worst case distribution over signals and relates it to the theory of asset pricing. Section 8 interprets our formulation and suggests modifications of it in terms of the multiple priors models of Epstein and Schneider (2003a) and Epstein and Schneider (2003b). Section 9 briefly relates our formulation to papers about reducing compound lotteries. Section 10 specializes our section 5 recursions to compute robust decision rules for the linear quadratic case, and appendix A reports useful computational tricks for this case. Section 11 concludes. Hansen and Sargent (2005) contains an extensive account of related literatures. An application to a decision problem with experimentation and learning about multiple submodels appears in Cogley, Colacito, Hansen, and Sargent (2005).

2 A control problem without model uncertainty

For $t \geq 0$, we partition a state vector as $x_t = \begin{bmatrix} y_t \\ z_t \end{bmatrix}$, where y_t is observed and z_t is not. A vector of s_t of observable signals is correlated with the hidden state z_t and is used by the decision maker to form beliefs about the hidden state. Let Z denote a space of admissible unobserved states, $\mathcal{Z}$ a corresponding sigma algebra of subsets of states, and λ a measure on the measurable space of hidden states $(Z, \mathcal{Z})$. Let S denote the space of signals, $\mathcal{S}$ a corresponding sigma algebra, and η a measure on the measurable space $(S, \mathcal{S})$ of signals.

Signals and states are determined by the transition functions

$$\begin{aligned} y_{t+1} &= \pi_y(s_{t+1}, y_t, a_t) & (1) \\ z_{t+1} &= \pi_z(x_t, a_t, w_{t+1}) & (2) \\ s_{t+1} &= \pi_s(x_t, a_t, w_{t+1}) & (3) \end{aligned}$$

where $\{w_{t+1} : t \geq 0\}$ is an i.i.d. sequence of random variables. Knowledge of y_0 and π_y allows us to construct y_t recursively from signals and actions. Substituting (3) into (1) gives the recursive evolution for the observable state in terms of next period's shock w_{t+1}:

$$y_{t+1} = \pi_y[\pi_s(x_t, a_t, w_{t+1}), y_t, a_t] \doteq \bar{\pi}_y(x_t, a_t, w_{t+1}) \tag{4}$$

Equations (2) and (3) determine a conditional density $\tau(z_{t+1}, s_{t+1}|x_t, a_t)$ relative to the product measure $\lambda \times \eta$.

Let $\{\mathcal{S}_t : t \geq 0\}$ denote a filtration, where $\mathcal{S}_t$ is generated by y_0, s_1, ..., s_t. We can apply Bayes' rule to τ to deduce a density q_t, relative to the measure λ, for z_t conditioned on information $\mathcal{S}_t$. Let $\{\mathcal{X}_t : t \geq 0\}$ be a larger filtration where $\mathcal{X}_t$ is generated by x_0, w_1, w_2, ..., w_t. The smallest sigma algebra generated by all states for $t \geq 0$ is $\mathcal{X}_\infty \doteq \bigvee_{t \geq 0} \mathcal{X}_t$; the smallest sigma algebra generated by all signals for $t \geq 0$ is $\mathcal{S}_\infty \doteq \bigvee_{t \geq 0} \mathcal{S}_t$. Let A denote a feasible set of actions, which we take to be a Borel set of some finite dimensional Euclidean space, and let $\mathcal{A}_t$ be the set of A-valued random vectors that are $\mathcal{S}_t$ measurable. Given the recursive construction of x_t in equation (1) - (2) and the informational constraint on action processes, x_t is $\mathcal{X}_t$ measurable and y_t is $\mathcal{S}_t$ measurable.

As a benchmark, consider the following decision problem under incomplete information about the state but complete confidence in the model (1), (2), (3):

Problem 2.1.

$$\max_{a_t \in \mathcal{A}_t : t \geq 0} E\left[\sum_{t=0}^{T} \beta^t U(x_t, a_t) | \mathcal{S}_0\right], \quad \beta \in (0, 1)$$

subject to (1), (2), and (3).

To make problem 2.1 recursive, use τ to construct two densities for the signal:

$$\begin{aligned} \kappa(s^*|y_t, z_t, a_t) &\doteq \int \tau(z^*, s^*|y_t, z_t, a_t) d\lambda(z^*) \\ \varsigma(s^*|y_t, q_t, a_t) &\doteq \int \kappa(s^*|y_t, z, a_t) q_t(z) d\lambda(z). \end{aligned} \tag{5}$$

By Bayes' rule, $q_{t+1}(z^*) = \frac{\int \tau(z^*, s_{t+1}|y_t, z, a_t) q_t(z) d\lambda(z)}{\varsigma(s_{t+1}|y_t, q_t, a_t)} \equiv \pi_q(s_{t+1}, y_t, q_t, a_t)$. In particular applications, π_q can be computed with methods that specialize Bayes' rule (*e.g.*, the Kalman filter or a discrete time version of the Wonham (1964) filter).

Take (y_t, q_t) as the state for a recursive formulation of problem 2.1. The transition law is (1) and

$$q_{t+1} = \pi_q(s_{t+1}, y_t, q_t, a_t). \tag{6}$$

Let $\pi = \begin{bmatrix} \pi_y \\ \pi_q \end{bmatrix}$. Then we can rewrite problem 2.1 in the alternative form:

Problem 2.2. *Choose a sequence of decision rules for a_t as functions of (y_t, q_t) for each $t \geq 0$ that maximizes*

$$E\left[\sum_{t=0}^{T} \beta^t U(x_t, a_t) | \mathcal{S}_0\right]$$

subject to (1), (6), a given density $q_0(z)$, and the density $\kappa(s_{t+1}|y_t, z_t, a_t)$. The Bellman equation for this problem is

$$W(y, q) = \max_{a \in A} \int \left\{U(x, a) + \beta \int W^* \left[\pi(s^*, y, q, a)\right] \kappa(s^*|y, z, a) d\eta(s^*)\right\} q(z) d\lambda(z). \tag{7}$$

In an infinite horizon version of problem 2.2, $W^* = W$.

2.1 Examples

Examples of problem 2.2 in economics include Jovanovic (1979), Jovanovic (1982), Jovanovic and Nyarko (1995), Jovanovic and Nyarko (1996), and Bergemann and Valimaki (1996). Examples from outside economics appear in Elliott, Aggoun, and Moore (1995). Problems that we are especially interested in are illustrated in the following four examples.

Example 2.3. *Model Uncertainty I: two submodels. Let the hidden state $z \in \{0, 1\}$ index a submodel. Let*

$$\begin{aligned} y_{t+1} &= s_{t+1} \\ z_{t+1} &= z_t \\ s_{t+1} &= \pi_s(y_t, z, a_t, w_{t+1}). \end{aligned} \tag{8}$$

The hidden state is time invariant. The decision maker has a prior probability $\text{Prob}(z = 0) = q$*. The third equation in (8) depicts two laws of motion. Cogley, Colacito, and Sargent (2005) and Cogley, Colacito, Hansen, and Sargent (2005) study the value of monetary policy experimentation in a model in which a is an inflation target and $\pi_s(y, z, a, w) = \bar{\pi}_y(y, z, a, w)$ for $z \in \{0, 1\}$ represent two submodels of inflation-unemployment dynamics.*

Example 2.4. *Model Uncertainty II: a continuum of submodels. The observable state y takes the two possible values $\{y_L, y_H\}$. Transition dynamics are still described by (8), but now there is a continuum of models indexed by the hidden state $z \in [0, 1] \times [0, 1]$ that stands for unknown values of two transition probabilities for an observed state variable y. Given z, we can use the third equation of (8) to represent a two state Markov chain on the observable state y (see Elliott, Aggoun, and Moore (1995)), $P = \begin{bmatrix} p_{11} & 1 - p_{11} \\ 1 - p_{22} & p_{22} \end{bmatrix}$, where $(p_{11}, p_{22}) = z$. The decision maker has a prior $f_0(p_{11}) g_0(p_{22})$ on z; f_0 and g_0 are beta distributions.*

Example 2.5. *Model Uncertainty III: A components model of income dynamics with an unknown fixed effect in labor income. The utility function $U(a_t)$ is a concave function of consumption a_t; y_{2t} is the level of financial assets, and $y_{1t} = s_t$ is observed labor income. The evolution equations are*

$$\begin{aligned} y_{1,t+1} &= s_{t+1} \\ y_{2,t+1} &= R[y_{2,t} + y_{1,t} - a_t] \\ z_{1,t+1} &= z_{1,t} \\ z_{2,t+1} &= \rho z_{2,t} + \sigma_1 w_{1,t+1} \\ s_{t+1} &= z_{1,t} + z_{2,t} + \sigma_2 w_{2,t+1} \end{aligned}$$

where $w_{t+1} \sim \mathcal{N}(0, I)$ is an i.i.d. bivariate Gaussian process, $R \leq \beta^{-1}$ is a gross return on financial assets $y_{2,t}$, $|\rho| < 1$, $z_{1,t}$ is an unobserved constant component of labor income, and $z_{2,t}$ is an unobserved serially correlated component of labor income. A decision maker has a prior q_0 over $(z_{1,0}, z_{2,0})$.

Example 2.6. *Estimation of drifting coefficients regression model. The utility function $U(x_t, a_t) = -L(z_t - a_t)$, where L is a loss function and a_t is a time-t estimator of the coefficient vector z_t. The evolution equation is*

$$\begin{aligned} y_{t+1} &= s_{t+1} \\ z_{t+1} &= \rho z_t + \sigma_1 w_{1,t+1} \\ s_{t+1} &= y_t \cdot z_t + \sigma_2 w_{2,t+1} \end{aligned}$$

where $w_{t+1} \sim \mathcal{N}(0, I)$ and there is a prior $q_0(z)$ on an initial set of coefficients.

2.2 Modified problems that distrust $\kappa(s^*|y, z, a)$ and $q(z)$

This paper studies modifications of problem 2.2 in which the decision maker wants a decision rule that is robust to possible misspecifications of equations (1)-(2). Bellman equation (7) indicates that the decision maker's concerns about misspecification of the stochastic structure can be focused on two aspects: the conditional distribution of next period's signals $\kappa(s^*|y, z, a)$ and the distribution over this period's value of the hidden state $q(z)$. We propose recursive formulations of a robust control problem that allow a decision maker to focus on either or both of these two aspects of his stochastic specification.

3 Using martingales to represent model misspecifications

Equations (1)-(2) induce a probability measure over $\mathcal{X}_t$ for $t \geq 0$. Hansen and Sargent (2005) use a nonnegative $\mathcal{X}_t$-measurable function M_t with $EM_t = 1$ to create a distorted probability measure that is absolutely continuous with respect to the probability measure over $\mathcal{X}_t$ generated by the model (1) - (2). The random variable M_t is a martingale under this baseline probability measure. Using M_t as a Radon-Nikodym derivative generates a distorted measure under which the expectation of a bounded $\mathcal{X}_t$-measurable random variable W_t is $\tilde{E}W_t \doteq EM_tW_t$. The *entropy* of the distortion at time t conditioned on date zero information is $E\left(M_t \log M_t|\mathcal{X}_0\right)$ or $E(M_t \log M_t|\mathcal{S}_0)$.

3.1 Recursive representations of distortions

It is convenient to factor a density F_t for an $\mathcal{X}_t$-measurable random variable F_t as $F_{t+1} = F_t f_{t+1}$ where f_{t+1} is a one-step ahead density conditioned on $\mathcal{X}_t$. It is useful to factor M_t in a similar way. Thus, to represent distortions recursively, take a nonnegative martingale $\{M_t : t \geq 0\}$ and form

$$m_{t+1} = \begin{cases} \frac{M_{t+1}}{M_t} & \text{if} \quad M_t > 0 \\ 1 & \text{if} \quad M_t = 0. \end{cases}$$

Then $M_{t+1} = m_{t+1}M_t$ and

$$M_t = M_0 \prod_{j=1}^{t} m_j. \tag{9}$$

The random variable M_0 has unconditional expectation equal to unity. By construction, m_{t+1} has date t conditional expectation equal to unity. For a bounded random variable W_{t+1} that is $\mathcal{X}_{t+1}$-measurable, the distorted conditional expectation implied by the martingale $\{M_t : t \geq 0\}$ is

$$\frac{E(M_{t+1}W_{t+1}|\mathcal{X}_t)}{E(M_{t+1}|\mathcal{X}_t)} = \frac{E(M_{t+1}W_{t+1}|\mathcal{X}_t)}{M_t} = E\left(m_{t+1}W_{t+1}|\mathcal{X}_t\right)$$

provided that $M_t > 0$. We use m_{t+1} to model distortions of the conditional probability distribution for $\mathcal{X}_{t+1}$ given $\mathcal{X}_t$. For each $t \geq 0$, construct the space $\mathcal{M}_{t+1}$ of all nonnegative, $\mathcal{X}_{t+1}$-measurable random variables m_{t+1} for which $E(m_{t+1}|\mathcal{X}_t) = 1$.

The conditional (on $\mathcal{X}_t$) relative entropy of a nonnegative random variable m_{t+1} in $\mathcal{M}_{t+1}$ is $\varepsilon_t^1(m_{t+1}) \doteq E\left(m_{t+1} \log m_{t+1}|\mathcal{X}_t\right)$.

3.2 Distorting likelihoods with hidden information

The random variable M_t is adapted to $\mathcal{X}_t$ and is a likelihood ratio for two probability distributions over $\mathcal{X}_t$. The $\mathcal{S}_t$-measurable random variable $G_t = E\left(M_t|\mathcal{S}_t\right)$ implies a likelihood ratio for the reduced information set $\mathcal{S}_t$; G_t assigns distorted expectations to $\mathcal{S}_t$-measurable random variables that agree with M_t, and $\{G_t : t \geq 0\}$ is a martingale adapted to $\{\mathcal{S}_t : t \geq 0\}$.

Define the $\mathcal{X}_t$-measurable random variable h_t by

$$h_t \doteq \begin{cases} \frac{M_t}{E(M_t|\mathcal{S}_t)} & \text{if} \quad E\left(M_t|\mathcal{S}_t\right) > 0 \\ 1 & \text{if} \quad E\left(M_t|\mathcal{S}_t\right) = 0 \end{cases}$$

and decompose M_t as

$$M_t = h_t G_t. \tag{10}$$

Decompose entropy as

$$E(M_t \log M_t|\mathcal{S}_0) = E\left[E\left(h_t \log h_t|\mathcal{S}_t\right) + G_t \log G_t|\mathcal{S}_0\right].$$

Define $\varepsilon_t^2(h_t) \doteq E\left(h_t \log h_t | \mathcal{S}_t\right)$ as the conditional (on $\mathcal{S}_t$) relative entropy.

We now have the tools to represent and measure misspecifications of the two components $\kappa(s^*|y,z,a)$ and $q(z)$ in (7). In (10), M_t distorts the probability distribution of $\mathcal{X}_t$, h_t distorts the probability of $\mathcal{X}_t$ conditioned on $\mathcal{S}_t$, G_t distorts the probability of $\mathcal{S}_t$, and m_{t+1} distorts the probability of $\mathcal{X}_{t+1}$ given $\mathcal{X}_t$. We use multiplication by m_{t+1} to distort κ and multiplication by h_t to distort q; and we use $\epsilon_t^1(m_{t+1})$ to measure m_{t+1} and $\epsilon_t^2(h_t)$ to measure h_t.

Section 4 uses these distortions to define two pairs of operators, then section 5 applies them to form counterparts to Bellman equation (7) that can be used to get decisions that are robust to these misspecifications.

4 Two pairs of operators

This section introduces two pairs of operators, $(\mathsf{R}_t^1, \mathsf{T}^1)$ and $(\mathsf{R}_t^2, \mathsf{T}^2)$. In section 5, we use the T^1 and T^2 operators to define recursions that induce robust decision rules.

4.1 R_t^1 and T^1

For $\theta > 0$, let W_{t+1} be an $\mathcal{X}_{t+1}$-measurable random variable for which $E\left[\exp\left(-\frac{W_{t+1}}{\theta}\right)|\mathcal{X}_t\right] < \infty$. Then define

$$
\begin{aligned}
\mathsf{R}_t^1\left(W_{t+1}|\theta\right) &= \min_{m_{t+1}\in\mathcal{M}_{t+1}} E\left(m_{t+1}W_{t+1}|\mathcal{X}_t\right) + \theta\varepsilon_t^1(m_{t+1}) \\
&= -\theta \log E\left[\exp\left(-\frac{W_{t+1}}{\theta}\right)|\mathcal{X}_t\right]. \qquad (11)
\end{aligned}
$$

The minimizing choice of m_{t+1} is

$$
m_{t+1}^* = \frac{\exp\left(-\frac{W_{t+1}}{\theta}\right)}{E\left[\exp\left(-\frac{W_{t+1}}{\theta}\right)|\mathcal{X}_t\right]}. \qquad (12)
$$

In the limiting case that sets the entropy penalty parameter $\theta = \infty$, $\mathsf{R}_t^1(W_{t+1}|\infty) = E(W_{t+1}|\mathcal{X}_t)$. Notice that this expectation can depend on the hidden state. When $\theta < \infty$, R_t^1 adjusts $E(W_{t+1}|\mathcal{X}_t)$ by using a worst-case belief about the probability distribution of $\mathcal{X}_{t+1}$ conditioned on $\mathcal{X}_t$ that is implied by the twisting factor (12). When the conditional moment restriction $E\left[\exp\left(-\frac{W_{t+1}}{\theta}\right)|\mathcal{X}_t\right] < \infty$ is not satisfied, we define R_t^1 to be $-\infty$ on the relevant conditioning events.

When the $\mathcal{X}_{t+1}$-measurable random variable W_{t+1} takes the special form $V(y_{t+1}, q_{t+1}, z_{t+1})$, the $\mathsf{R}_t^1(\cdot|\theta)$ operator defined in (11) implies another operator:

$$
(\mathsf{T}^1 V|\theta)(y,q,z,a) = -\theta \log \int \exp\left(-\frac{V\left[\pi(s^*,y,q,a),z^*\right]}{\theta}\right)\tau(z^*,s^*|y,z,a)d\lambda(z^*)d\eta(s^*).
$$

The transformation T^1 maps a value function that depends on next period's state (y^*,q^*,z^*) into a risk-adjusted value function that depends on (y,q,z,a). Associated with this *risk* *adjustment* is a worst-case distortion in the transition dynamics for the state and signal process. Let ϕ denote a nonnegative density function defined over (z^*, s^*) satisfying

$$\int \phi(z^*, s^*)\tau(z^*, s^*|y, z, a)d\lambda(z^*)d\eta(s^*) = 1. \tag{13}$$

The corresponding entropy measure is:

$$\int \log[\phi(z^*, s^*)]\phi(z^*, s^*)\tau(z^*, s^*|y, z, a)d\lambda(z^*)d\eta(s^*) = 1.$$

In our recursive formulation, we think of ϕ as a possibly infinite dimensional control vector (a density function) and consider the minimization problem:

$$\min_{\phi \geq 0} \int \left(V\left[\pi(s^*, y, q, a), z^*\right] + \theta_1 \log[\phi(z^*, s^*)]\right) \phi(z^*, s^*)\tau(z^*, s^*|y, z, a)d\lambda(z^*)d\eta(s^*)$$

subject to (13). The associated worst-case density conditioned on $\mathcal{X}_t$ is $\phi_t(z^*, s^*)\tau(z^*, s^*|x_t, a_t)$ where

$$\phi_t(z^*, s^*) = \frac{\exp\left(-\frac{V[\pi(s^*, y_t, q_t, a_t), z^*]}{\theta}\right)}{E\left[\exp\left(-\frac{V[\pi(s_{t+1}, y_t, q_t, a_t), z_{t+1}]}{\theta}\right)|\mathcal{X}_t\right]}. \tag{14}$$

4.2 R_t^2 and T^2

For $\theta > 0$, let $\hat{W}_t$ be an $\mathcal{X}_t$-measurable function for which $E\left[\exp\left(-\frac{\hat{W}_t}{\theta}\right)|\mathcal{S}_t\right] < \infty$. Then define

$$\begin{aligned} \mathsf{R}_t^2\left(\hat{W}_t|\theta\right) &= \min_{h_t \in \mathcal{H}_t} E\left(h_t \hat{W}_t|\mathcal{S}_t\right) + \theta\varepsilon_t^2(h_t) \\ &= -\theta \log E\left[\exp\left(-\frac{\hat{W}_t}{\theta}\right)|\mathcal{S}_t\right]. \end{aligned} \tag{15}$$

The minimizing choice of h_t is

$$h_t^* = \frac{\exp\left(-\frac{\hat{W}_t}{\theta}\right)}{E\left[\exp\left(-\frac{\hat{W}_t}{\theta}\right)|\mathcal{S}_t\right]}.$$

When an $\mathcal{X}_t$-measurable function has the special form $\hat{W}_t = \hat{V}(y_t, q_t, z_t, a_t)$, R_t^2 given by (15) implies an operator

$$(\mathsf{T}^2\hat{V}|\theta)(y, q, a) = -\theta \log \int \exp\left[-\frac{\hat{V}(y, q, z, a)}{\theta}\right] q(z)d\lambda(z).$$

The associated minimization problem is:

$$\min_{\psi \geq 0} \int \left[\hat{V}(y, q, z, a) + \theta \log \psi(z)\right] \psi(z)q(z)d\lambda(z)$$

subject to (16), where $\psi(z)$ is a relative density that satisfies:

$$\int \psi(z)q(z)d\lambda(z) = 1 \tag{16}$$

and the entropy measure is

$$\int [\log \psi(z)]\psi(z)q(z)d\lambda(z).$$

The optimized density conditioned on $\mathcal{S}_t$ is $\psi_t(z)q_t(z)$, where

$$\psi_t(z) = \frac{\exp\left(-\frac{\hat{V}(y_t,q_t,z,a_t)}{\theta}\right)}{E\left[\exp\left(-\frac{\hat{V}(y_t,q_t,z,a_t)}{\theta}\right)|\mathcal{S}_t\right]}. \tag{17}$$

5 Control problems with model uncertainty

We propose robust control problems that take $q_t(z)$ as the decision maker's state variable for summarizing the history of signals. The decision maker's model includes the law of motion (6) for q (Bayes' law) under the approximating model (1), (2), (3). Two recursions that generalize Bellman equation (7) express alternative views about the decision maker's fear of misspecification. A first recursion works with value functions that include the hidden state z as a state variable. Let

$$\check{W}(y,q,z) = U(x,a) + E\left\{\beta \check{W}^*[\pi(s^*,y,q,a),z^*]\big|x,q\right\}, \tag{18}$$

where the action a solves:

$$W(y,q) = \max_a E\left[U(x,a) + E\Big\{\beta\check{W}^*[\pi(s^*,y,q,a),z^*]\big|x,q,a\Big\}\Big|y,q,a\right]. \tag{19}$$

The value function $\check{W}$ depends on the hidden state z, whereas the value function W in (7) does not. A second recursion modifies the ordinary Bellman equation (7), which we can express as:

$$W(y,q) = \max_a E\left[U(x,a) + E\Big\{\beta W^*[\pi(s^*,y,q,a)]\big|x,q,a\Big\}\Big|y,q,a\right]. \tag{20}$$

Although they use different value functions, without concerns about model misspecification, formulations (18)-(19) and (20) imply identical control laws. Furthermore, $W(y,q)$ obeys (19), by virtue of the law of iterated expectations. Because Bellman equation (20) is computationally more convenient, the pair (18)-(19) is not used in the standard problem without a concern for robustness. However, with a concern about robustness, a counterpart to (18)-(19) becomes useful when the decision maker wants to explore distortions of the joint conditional distribution $\tau(s^*,z^*|y,z,a)$.[2] Distinct formulations emerge when we

[2]Another way to express his concerns is that in this case the decision maker fears that (2) and (3) are both misspecified.

replace the conditional expectation $E(\cdot|y,q,a)$ with $\mathsf{T}^2(\cdot|\theta_2)$ and the conditional expectation $E(\cdot|x,q,a)$ with $\mathsf{T}^1(\cdot|\theta_1)$ in the above sets of recursions. When $\theta_1 = \theta_2 = +\infty$, (18)-(19) or (20) lead to value functions and decision rules equivalent to those from either (18)-(19) or (20). When $\theta_1 < +\infty$ and $\theta_2 < +\infty$, they differ because they take different views about which conditional distributions the malevolent player wants to distort.

5.0.1 Which conditional distributions to distort?

The approximating model (1), (2), (3) makes both tomorrow's signal s^* and tomorrow's state z^* functions of x. When tomorrow's value function depends on s^* but not on z^*, the minimizing player chooses to distort only $\kappa(s^*|y,z,a)$, which amounts to being concerned about misspecified models only for the evolution equation (3) for the signal and not (2) for the hidden state. Such a continuation value function imparts no additional incentive to distort the evolution equation (2) of z^* conditioned on s^* and x.[3] A continuation value that depends on s^* but not on z^* thus imparts concerns about a limited array of distortions that ignore possible misspecification of the z^* evolution (2). Therefore, when we want to direct the maximizing agent's concerns about misspecification onto the conditional distribution $\kappa(s^*|y,z,a)$, we should form a current period value that depends only on the history of the signal and of the observed state. We do this in recursion (23) below.

In some situations, we want to extend the maximizing player's concerns about misspecification to the joint distribution $\tau(z^*,s^*|y,z,a)$ of z^* and s^*. We can do this by making tomorrow's value function for the minimizing player also depend on z^*. This will prompt the minimizing agent to distort the joint distribution $\tau(z^*,s^*|y,z,a)$ of (z^*,s^*). In recursions (21)-(22) below, we form a continuation value function that depends on z^* and that extends recursions (18), (19) to incorporate concerns about misspecification of (2).

Thus, (21)-(22) below will induce the minimizing player to distort the distribution of z^* conditional on (s^*,x,a), while the formulation in (23) will not.

5.1 Value function depends on (x,q)

By defining a value function that depends on the hidden state, we focus the decision maker's attention on misspecification of the joint conditional distribution $\tau(z^*,s^*|y,z,a)$ of (s^*,z^*). We modify recursions (18)-(19) by updating a value function according to

$$\check{W}(y,q,z) = U(x,a) + \mathsf{T}^1\left[\beta\check{W}^*(y^*,q^*,z^*)|\theta_1\right](x,q,a) \tag{21}$$

after choosing an action according to

$$\max_a \mathsf{T}^2\Big\{U(x,a) + \mathsf{T}^1\Big[\beta\check{W}^*(y^*,q^*,z^*)|\theta_1\Big](x,q,a)\Big|\theta_2\Big\}(y,q,a), \tag{22}$$

for $\theta_1 \geq \underline{\theta}_1$, $\theta_2 \geq \underline{\theta}_2(\theta_1)$ for $\underline{\theta}_1, \underline{\theta}_2$ that make the problems well posed.[4] Updating the value function by recursion (21) makes it depend on (x,q), while using (22) to guide decisions makes

[3] Dependence between (s^*,z^*) conditioned on x under the approximating model means that in the process of distorting s^* conditioned on (x,a), the minimizing player may indirectly distort the distribution of z^* conditioned on (x,a). But he does not distort the distribution of z^* conditioned on (s^*,x,a)

[4] Limits on θ_1 and θ_2 are typically needed to make the outcomes of the T^1 and T^2 operators be finite.

actions depend only on the observable state (y, q). Thus, continuation value $\check{W}$ depends on unobserved states, but actions do not. To retain the dependence of the continuation value on z, (21) refrains from using the T^2 transformation when up-dating continuation values. The fixed point of (21)-(22) is the value function for an infinite horizon problem. For the finite horizon counterpart, we begin with a terminal value function and view the right side of (21) as mapping next period's value function into the current period value function.

5.2 Value function depends on (y, q)

To focus attention on misspecifications of the conditional distribution $\kappa(s^*|y, z, a)$, we want the minimizing player's value function to depend only on the reduced information encoded in (y, q). For this purpose, we use the following counterpart to recursion (20):

$$W(y, q) = \max_a \mathsf{T}^2 \left(U(x, a) + \mathsf{T}^1 \left[\beta W^*(y^*, q^*) | \theta_1 \right] (x, q, a) \middle| \theta_2 \right) (y, q, a) \tag{23}$$

for $\theta_1 \geq \underline{\theta}_1$ and $\theta_2 \geq \underline{\theta}_2(\theta_1)$. Although z^* is excluded from the value function W^*, z may help predict the observable state y^* or it may enter directly into the current period reward function, so application of the operator T^1 creates a value function that depends on (x, q, a), including the hidden state z. Since the malevolent agent observes z, he can distort the dynamics for the observable state conditioned on z via the T^1 operator. Subsequent application of T^2 gives a value function that depends on (y, q, a), but not z; T^2 distorts the hidden state distribution. The decision rule sets action a as a function of (y, q). The fixed point of Bellman equation (23) gives the value function for an infinite horizon problem. For finite horizon problems, we iterate on the mapping defined by the right side of (23), beginning with a known terminal value function. Recursion (23) extends the recursive formulation of risk-sensitivity with discounting advocated by Hansen and Sargent (1995) to situations with a hidden state.

5.3 Advantages of our specification

We take the distribution $q_t(z)$ as a state variable and explore misspecifications of it. An alternative way to describe a decision maker's fears of misspecification would be to perturb the evolution equation for the hidden state (1) directly. Doing that would complicate the problem substantially by requiring us to solve a filtering problem for each perturbation of (1). Our formulation avoids multiple filtering problems by solving one and only one filtering problem under the approximating model. The transition law π_q for $q(z)$ in (6) becomes a component of the approximating model.

When $\theta_1 = +\infty$ but $\theta_2 < +\infty$, the decision maker trusts the signal dynamics $\kappa(s^*|y, z, a)$ but distrusts $q(z)$. When $\theta_2 = +\infty$ but $\theta_1 < +\infty$, the situation is reversed. The two-θ formulation thus allows the decision maker to distinguish his suspicions of these two aspects of the model. Before saying more about the two-θ formulation, the next section explores some ramifications of the special case in which $\theta_1 = \theta_2$ and how it compares to the single θ specification that prevails in versions of our decision problem under commitment.

6 The $\theta_1 = \theta_2$ case

For the purpose of studying intertemporal consistency and other features of the associated worst case models, it is interesting to compare the outcomes of recursions (21)-(22) or (23) with the decision rule and worst case model described by Hansen and Sargent (2005) in which at time 0 the maximizing and minimizing players in a zero-sum game commit to a sequence of decision rules and a single worst case model, respectively. Because there is a single robustness parameter θ in this "commitment model", it is natural to make this comparison for the special case in which $\theta_1 = \theta_2$.

6.1 A composite operator $\mathsf{T}^2 \circ \mathsf{T}^1$ when $\theta_1 = \theta_2$

When a common value of θ appears in the two operators, the sequential application $\mathsf{T}^2\mathsf{T}^1$ can be replaced by a single operator:

$$\begin{aligned}\mathsf{T}^2 \circ \mathsf{T}^1 \Big[U(x,a) \quad &+ \quad \beta W(y^*, q^*)\Big](y,q,a) \\ &= \quad -\theta \log \int \exp\left(-\frac{U(x,a) + \beta W\left[\pi(s^*, y, q, a)\right]}{\theta}\right) \kappa(s^*|y,z,a)q(z)d\eta(s^*)d\lambda(z).\end{aligned}$$

This operator is the outcome of a portmanteau minimization problem where the minimization is over a single relative density $\varphi(s^*, z) \geq 0$ that satisfies[5]

$$\int \varphi(s^*, z)\kappa(s^*|y,z,a)q(z)d\eta(s^*)d\lambda(z) = 1,$$

where φ is related to ϕ and ψ defined in (13) and (16) by

$$\varphi(s^*, z) = \int \phi(z^*, s^*|z)\psi(z)q^*(z^*)d\lambda(z^*),$$

where this notation emphasizes that the choice of ϕ can depend on z. The entropy measure for φ is

$$\int [\log \varphi(s^*, z)]\varphi(s^*, z)\kappa(s^*|y,z,a)q(z)d\eta(s^*)d\lambda(z),$$

and the minimizing composite distortion φ to the joint density of (s^*, z) given $\mathcal{S}_t$ is

$$\varphi_t(s^*, z) = \frac{\exp\left(-\frac{U(y_t,z,a_t) + \beta W[\pi(s^*, y_t, q_t, a_t)]}{\theta}\right)}{E\left[\exp\left(-\frac{U(y_t,z,a_t) + \beta W[\pi(s_{t+1}, y_t, q_t, a_t)]}{\theta}\right)|\mathcal{S}_t\right]}. \tag{24}$$

6.2 Special case $U(x,a) = \hat{U}(y,a)$

When $U(x,a) = \hat{U}(y,a)$, the current period utility drops out of formula (24) for the worst-case distortion to the distribution, and it suffices to integrate with respect to the distribution

[5]For comparison, recall that applying T^1 and T^2 separately amounts to minimizing over separate relative densities ϕ and ψ.

ς that we constructed in (5) by averaging κ over the distribution of the hidden state. Probabilities of future signals compounded by the hidden state are simply averaged out using the state density under the benchmark model, a reduction of a compound lottery that would not be possible if different values of θ were to occur in the two operators.

To understand these claims, we deduce a useful representation of $\varepsilon_t(m_{t+1}, h_t)$:

$$\min_{m_{t+1}\in\mathcal{M}_t, h_t\in\mathcal{H}_t} E\left[h_t\varepsilon_t^1(m_{t+1})|\mathcal{S}_t\right] + \varepsilon_t^2(h_t)$$

subject to $E(m_{t+1}h_t|\mathcal{S}_{t+1}) = g_{t+1}$, where $E(g_{t+1}|\mathcal{S}_t) = 1$, a constraint that we impose because our aim is to distort expectations of $\mathcal{S}_{t+1}$-measurable random variables given current information $\mathcal{S}_t$. The minimizer is

$$m_{t+1}^* = \begin{cases} \frac{g_{t+1}}{E(g_{t+1}|\mathcal{X}_t)} & \text{if} \quad E(g_{t+1}|\mathcal{X}_t) > 0 \\ 0 & \text{if} \quad E(g_{t+1}|\mathcal{X}_t) = 0 \end{cases}$$

and $h_t^* = E(g_{t+1}|\mathcal{X}_t)$. Therefore, $m_{t+1}^* h_t^* = g_{t+1}$ and the minimized value of the objective is

$$\varepsilon_t(m_{t+1}^*, h_t^*) = E\left[g_{t+1}\log(g_{t+1})|\mathcal{S}_t\right] \equiv \tilde{\epsilon}_t(g_{t+1}). \tag{25}$$

Thus, in distorting continuation values that are $\mathcal{S}_t$-measurable, it suffices to use entropy measure $\tilde{\epsilon}_t$ defined in (25) and to explore distortions to the conditional probability of $\mathcal{S}_{t+1}$-measurable events given $\mathcal{S}_t$. This is precisely what the g_{t+1} random variable accomplishes. The g_{t+1} associated with $\mathsf{T}^2\mathsf{T}^1$ in the special case in which $U(x,a) = \hat{U}(y,a)$ implies a distortion ϕ_t in equation (14) that depends on s^* alone. The iterated operator $\mathsf{T}^2\mathsf{T}^1$ can be regarded as a single risk-sensitivity operator analogous to T^1:

$$\begin{aligned} \mathsf{T}^2\mathsf{T}^1\Big[\hat{U}(y,a) \quad &+ \quad \beta W^*(y^*, q^*)\Big](y,q,a) \\ &= \quad \hat{U}(y,a) - \theta\log\int\exp\left(-\frac{\beta W^*(\pi(s^*,y,q,a))}{\theta}\right)\varsigma(s^*|y,q,a)d\eta(s^*). \end{aligned} \tag{26}$$

In section A.4 of appendix A, we describe how to compute this operator for linear quadratic problems.

6.3 Comparison with outcomes under commitment

Among the outcomes of iterations on the recursions (21)-(22) or (23) of section 5 are time-invariant functions that map (y_t, q_t) into a pair of nonnegative random variables (m_{t+1}, h_t). For the moment, ignore the distortion h_t and focus exclusively on m_{t+1}. Through (9), the time-invariant rule for m_{t+1} can be used to a construct a martingale $\{M_t : t \geq 0\}$. This martingale implies a limiting probability measure on $\mathcal{X}_\infty = \vee_{t\geq 0}\mathcal{X}_t$ via the Kolmogorov extension theorem. The implied probability measure on $\mathcal{X}_\infty$ will typically not be absolutely continuous over the entire collection of limiting events in $\mathcal{X}_\infty$. Although the martingale converges almost surely by virtue of Doob's martingale convergence theorem, in the absence of this absolute continuity, the limiting random variable will not have unit expectation. This means that concerns about robustness persist in a way that they don't in a class of robust control problems under commitment that are studied, for example, by Whittle (1990) and Hansen and Sargent (2005).[6]

6.3.1 A problem under commitment and absolute continuity

Let M_∞ be a nonnegative random variable that is measurable with respect to $\mathcal{X}_\infty$, with $E(M_\infty|\mathcal{S}_0) = 1$. For a given action process $\{a_t : t \geq 0\}$ adapted to $\{\mathcal{X}_t : t \geq 0\}$, let $W_\infty \doteq \sum_{t=0}^{\infty} \beta^t U(x_t, a_t)$ subject to (1)-(2). Suppose that $\theta > 0$ is such that $E\left[\exp\left(-\frac{1}{\theta}W_\infty\right)|\mathcal{S}_0\right] < \infty$. Then

$$\begin{aligned} \mathsf{R}^1_\infty(W_\infty) &\doteq \min_{M_\infty \geq 0, E(M_\infty|\mathcal{S}_0)=1} E(M_\infty W_\infty|\mathcal{S}_0) + \theta E(M_\infty \log M_\infty|\mathcal{S}_0) && (28) \\ &= -\theta \log E\left[\exp\left(-\frac{1}{\theta}W_\infty\right)|\mathcal{S}_0\right]. && (29) \end{aligned}$$

This static problem has minimizer $M^*_\infty = \frac{\exp\left(-\frac{1}{\theta}W_\infty\right)}{E\left[\exp\left(-\frac{1}{\theta}W_\infty\right)|\mathcal{S}_0\right]}$ that implies a martingale $M^*_t = E\left(M^*_\infty|\mathcal{X}_t\right)$.[7] Control theory interprets (29) as a risk-sensitive adjustment of the criterion W_∞ (*e.g.*, see Whittle (1990)) and gets decisions that are robust to misspecifications by solving

$$\max_{a_t \in \mathcal{A}_t, t \geq 0} -\theta \log E\left[\exp\left(-\frac{1}{\theta}W_\infty\right)\Big|\mathcal{S}_0\right].$$

In a closely related setting, Whittle (1990) obtained time-varying decision rules for a_t that converge to ones that ignore concerns about robustness (i.e., those computed with $\theta = +\infty$).

The dissipation of concerns about robustness in this commitment problem is attributable to setting $\beta \in (0, 1)$ while using the undiscounted form of entropy in the criterion function (28). Those features lead to the existence of a well defined limiting random variable M_∞ with expectation unity (conditioned on $\mathcal{S}_0$), which means that tail events that are assigned probability zero under the approximating model are also assigned probability zero under the distorted model.[8]

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[6] The product decomposition (9) of M_t implies an additive decomposition of entropy:

$$E\left(M_t \log M_t|\mathcal{S}_0\right) - E\left(M_0 \log M_0|\mathcal{S}_0\right) = \sum_{j=0}^{t-1} E\left[M_j E\left(m_{j+1} \log m_{j+1}|\mathcal{X}_j\right)|\mathcal{S}_0\right]. \quad (27)$$

Setting $E(M_0|\mathcal{S}_0) = 1$ means that we distort probabilities conditioned on $\mathcal{S}_0$.

[7] See Dupuis and Ellis (1997). While robust control problems are often formulated as deterministic problems, here we follow Petersen, James, and Dupuis (2000) by studying a stochastic version with a relative entropy penalty.

[8] Because all terms on the right side of (27) are nonnegative, the sequence

$$\sum_{j=0}^{t-1} M_{j-1} E\left(m_j \log m_j|\mathcal{X}_{j-1}\right)$$

is increasing. Therefore, it has a limit that might be $+\infty$ with positive probability. Thus, $\lim_{t\to\infty} E(M_t \log M_t|\mathcal{S}_0)$ converges. Hansen and Sargent (2005) show that when this limit is finite almost surely, the martingale sequence $\{M_t : t \geq 0\}$ converges in the sense that $\lim_{t\to\infty} E\left(\ |M_t - M_\infty|\ |\mathcal{S}_0\right) = 0$, where M_∞ is measurable with respect to $\mathcal{X}_\infty \doteq \bigvee_{t=0}^{\infty} \mathcal{X}_t$. The limiting random variable M_∞ can be used to

6.3.2 Persistence of robustness concerns without commitment

In our recursive formulations (21)-(22) and (23) of section 5, the failure of the worst-case nonnegative martingale $\{M_t : t \geq 0\}$ to converge to a limit with expectation one (conditioned on $\mathcal{S}_0$) implies that the distorted probability distribution on $\mathcal{X}_\infty$ is not absolutely continuous with respect to the probability distribution associated with the approximating model. This feature sustains enduring concerns about robustness and permits time-invariant robust decision rules, in contrast to the outcomes with discounting in Whittle (1990) and Hansen and Sargent (2005), for example. For settings with a fully observed state vector, Hansen and Sargent (1995) and Hansen, Sargent, Turmuhambetova, and Williams (2004) formulated recursive problems that yielded time-invariant decision rules and enduring concerns about robustness by appropriately discounting entropy. The present paper extends these recursive formulations to problems with unobserved states.

6.4 Dynamic inconsistency of worst-case probabilities about hidden states

This section links robust control theory to recursive models of uncertainty aversion by exploring aspects of the worst case probability models that emerge from the recursions defined in section 5. Except in a special case that we describe in subsection 6.6, those recursions achieve dynamic consistency of decisions by sacrificing dynamic consistency of beliefs about hidden state variables. We briefly explore how this happens. Until we get to the special case analyzed in subsection 6.6, the arguments of this subsection will also apply to the general case in which $\theta_1 \neq \theta_2$.

Problems (11) and (15) that define R_t^1 and R_t^2, respectively, imply worst-case probability distributions that we express as a pair of Radon-Nikodym derivatives (m_{t+1}^*, h_t^*). The positive random variable m_{t+1}^* distorts the distribution of $\mathcal{X}_{t+1}$ conditioned on $\mathcal{X}_t$ and the positive random variable h_t^* distorts the distribution of events in $\mathcal{X}_t$ conditioned on $\mathcal{S}_t$. Are these probability distortions consistent with next period's distortion h_{t+1}^*? Not necessarily, because we have not imposed the pertinent consistency condition on these beliefs.

6.5 A belief consistency condition

To deduce a sufficient condition for consistency, recall that the implied $\{M_{t+1}^* : t \geq 0\}$ should be a martingale. Decompose M_{t+1}^* in two ways:

$$M_{t+1}^* = m_{t+1}^* h_t^* G_t^* = h_{t+1}^* G_{t+1}^*.$$

These equations involve G_{t+1}^* and G_t^*, both of which we have ignored in the recursive formulation of section 5. Taking expectations of $m_{t+1}^* h_t^* G_t^* = h_{t+1} G_{t+1}^*$ conditioned on $\mathcal{S}_{t+1}$ yields

$$G_t^* E\left(m_{t+1}^* h_t^* | \mathcal{S}_{t+1}\right) = G_{t+1}^*.$$

Thus,

$$g_{t+1}^* = E\left(m_{t+1}^* h_t^* | \mathcal{S}_{t+1}\right)$$

construct a probability measure on $\mathcal{X}_\infty$ that is absolutely continuous with respect to the probability measure associated with the approximating model. Moreover, $M_t = E(M_\infty | \mathcal{X}_t)$.

is the implied multiplicative increment for the candidate martingale $\{G_t^* : t \geq 0\}$ adapted to the signal filtration. Moreover,

Claim 6.1. *A sufficient condition for the distorted beliefs to be consistent is that the process $\{h_t^* : t \geq 0\}$ should satisfy:*

$$h_{t+1}^* = \begin{cases} \frac{m_{t+1}^* h_t^*}{E\left(m_{t+1}^* h_t^* | \mathcal{S}_{t+1}\right)} & \text{if} \quad E\left(m_{t+1}^* h_t^* | \mathcal{S}_{t+1}\right) > 0 \\ 1 & \text{if} \quad E\left(m_{t+1}^* h_t^* | \mathcal{S}_{t+1}\right) = 0. \end{cases} \tag{30}$$

This condition is necessary if $G_{t+1}^* > 0$.[9]

The robust control problem under commitment analyzed by Hansen and Sargent (2005) satisfies condition (30) by construction: at time 0 a single minimizing player chooses a pair (m_{t+1}^*, h_t^*) that implies next period's h_{t+1}^*. However, in the recursive games defined in the recursions (21)-(22) and (23) in section 5, the date t minimizing agent does not have to respect this constraint. A specification of h_{t+1}^* gives one distortion of the distribution of the hidden state (conditioned on $\mathcal{S}_{t+1}$) and the pair (m_{t+1}^*, h_t^*) gives another. We do not require that these agree, and, in particular, do not require that the probabilities of events in $\mathcal{X}_t$ be distorted in the same ways by the date t determined worst-case distribution (conditioned on $\mathcal{S}_{t+1}$) and the date $t+1$ worst-case distribution (conditioned on $\mathcal{S}_{t+1}$).

A conflict can arise between these worst-case distributions because choosing an action is naturally forward-looking, while estimation of z is backward looking. Dynamic inconsistency of *any* kind is a symptom of *conflicts* among the interests of different decision makers, and that is the case here. The two-player games that define the evaluation of future prospects (T^1) and estimation of the current position of the system (T^2) embody different orientations – T^1 looking to the *future*, T^2 focusing on an *historical* record of signals.

The inconsistency of the worst-case beliefs pertains only to the decision maker's opinions about the hidden state. If we ignore hidden states and focus on signals, we can assemble a consistent distorted signal distribution by constructing $g_{t+1}^* = E\left(m_{t+1}^* h_t^* | \mathcal{S}_{t+1}\right)$ and noting that $E\left(g_{t+1}^* | \mathcal{S}_t\right) = 1$, so that g_{t+1}^* is the implied one-period distortion in the signal distribution. We can construct a distorted probability distribution over events in $\mathcal{S}_{t+1}$ by using

$$G_{t+1}^* = \prod_{j=1}^{t+1} g_j^*. \tag{31}$$

Under this interpretation, the pair (m_{t+1}^*, h_t^*) is only a device to construct g_{t+1}^*. When the objective function U does not depend directly on the hidden state vector z, as is true in many economic problems, the consistent set of distorted probabilities defined by (31) describes the events that directly influence decisions.

[9] This consistency condition arguably could be relaxed for the two player game underlying (23). Although we allow m_{t+1} to depend on the signal s_{t+1} and the hidden state z_{t+1}, the minimizing solution associated with recursions (23) depends only on the signal s_{t+1}. Thus we could instead constrain the minimizing agent in his or her choice of m_{t+1} and introduce a random variable $\tilde{m}_{t+1}$ that distorts the probability distribution of z_{t+1} conditioned on s_{t+1} and $\mathcal{X}_t$. A weaker modified consistency requirement is that

$$h_{t+1}^* = \frac{\tilde{m}_{t+1} m_{t+1}^* h_t^*}{E\left(\tilde{m}_{t+1} m_{t+1}^* h_t^* | \mathcal{S}_{t+1}\right)}$$

for some $\tilde{m}_{t+1}$ with expectation equal to one conditioned on s_{t+1} and $\mathcal{X}_t$.

6.6 Discounting and preferences influenced by hidden states are the source of intertemporal inconsistency

If $\beta = 1$ and if $U(x, a)$ does not depend on the hidden state, we can show that the distortions (m_{t+1}, h_t) implied by our recursions satisfy the restriction of Claim 6.1 and so are temporally consistent. Therefore, in this special case, the recursive games imply the same decisions and worst case distortions as the game under commitment analyzed by Hansen and Sargent (2005). For simplicity, suppose that we fix an action process $\{a_t : t \geq 0\}$ and focus exclusively on the assignment of distorted probabilities. Let $\{W_t : t \geq 0\}$ denote the process of continuation values determined recursively and supported by choices of worst-case models.

Consider two operators R_t^1 and R_t^2 with a common θ. The operator R_t^1 implies a worst-case distribution for $\mathcal{X}_{t+1}$ conditioned on $\mathcal{X}_t$ with density proportional to:

$$m_{t+1}^* = \frac{\exp\left(-\frac{W_{t+1}}{\theta}\right)}{E\left[\exp\left(-\frac{W_{t+1}}{\theta}\right)|\mathcal{X}_t\right]}.$$

The operator R_t^2 implies a worst-case model for the probability of $\mathcal{X}_t$ conditioned on $\mathcal{S}_t$ with density:

$$h_t^* = \frac{E\left[\exp\left(-\frac{W_{t+1}}{\theta}\right)|\mathcal{X}_t\right]}{E\left[\exp\left(-\frac{W_{t+1}}{\theta}\right)|\mathcal{S}_t\right]}.$$

Combining the distortions gives

$$m_{t+1}^* h_t^* = \frac{\exp\left(-\frac{W_{t+1}}{\theta}\right)}{E\left[\exp\left(-\frac{W_{t+1}}{\theta}\right)|\mathcal{S}_t\right]}.$$

To establish temporal consistency, from Claim 6.1 we must show that

$$h_{t+1}^* = \frac{\exp\left(-\frac{W_{t+1}}{\theta}\right)}{E\left[\exp\left(-\frac{W_{t+1}}{\theta}\right)|\mathcal{S}_{t+1}\right]}$$

where

$$h_{t+1}^* \doteq \frac{E\left[\exp\left(-\frac{W_{t+2}}{\theta}\right)|\mathcal{X}_t\right]}{E\left[\exp\left(-\frac{W_{t+2}}{\theta}\right)|\mathcal{S}_t\right]}.$$

This relation is true when $\beta = 1$ and U does not depend on the hidden state z. To accommodate $\beta = 1$, we shift from an infinite horizon problem to a finite horizon problem with a terminal value function. From value recursion (21) and the representation of R_{t+1}^1 in (11),

$$\exp\left(-\frac{W_{t+1}}{\theta}\right) \propto E\left[\exp\left(-\frac{W_{t+2}}{\theta}\right)|\mathcal{X}_{t+1}\right],$$

where the proportionality factor is $\mathcal{S}_{t+1}$ measurable. The consistency requirement for h^*_{t+1} is therefore satisfied.

The preceding argument isolates the role that discounting plays in delivering the time inconsistency of worst case beliefs over the hidden state. Heuristically, the games defined by the recursions (21)-(22) or (23) give intertemporal inconsistency when $\beta < 1$ because the decision maker discounts both current period returns and current period increments to entropy; while in the commitment problem analyzed in Hansen and Sargent (2005), the decision maker discounts current period returns but not current period increments to entropy.

7 Implied worst case model of signal distortion

The martingale (relative to $\mathcal{S}_t$) increment $g_{t+1} = E\left(m_{t+1}h_t|\mathcal{S}_t\right)$ distorts the distribution of the date $t+1$ signal given information $\mathcal{S}_t$ generated by current and past signals. For the following three reasons, it is interesting to construct an implied g^*_{t+1} from the m^*_{t+1} associated with R^1_t or T^1 and the h^*_t associated with R^2_t or T^2.

First, actions depend only on signal histories. Hidden states are used either to depict the underlying uncertainty or to help represent preferences. However, agents cannot take actions contingent on these hidden states, only on the signal histories.

Second, in decentralized economies, asset prices can be characterized by stochastic discount factors that equal the intertemporal marginal rates of substitution of unconstrained investors and that depend on the distorted probabilities that investors use to value contingent claims. Since contingent claims to consumption can depend only on signal histories (and not on hidden states), the distortion to the signal distribution is the twist to asset pricing that is contributed by investors' concerns about model misspecification. In particular, under the approximating model, $\frac{g_{t+1}}{E[g_{t+1}|\mathcal{S}_t]}$ becomes a multiplicative adjustment to the ordinary stochastic discount factor for a representative agent (e.g., see Hansen, Sargent, and Tallarini (1999)). It follows that the temporal inconsistency of worst case beliefs discussed in section 6.4 does not impede appealing to standard results on the recursive structure of asset pricing in settings with complete markets.[10]

Third, Anderson, Hansen, and Sargent (2003) found it useful to characterize detection probabilities using relative entropy and an alternative measure of entropy due to Chernoff (1952). Chernoff (1952) showed how detection error probabilities for competing models give a way to measure model discrepancy. Models are close when they are hard to distinguish with historical data. Because signal histories contain all data that are available to a decision maker, the measured entropy from distorting the signal distribution is pertinent for statistical discrimination. These lead us to measure either $E\left(g^*_{t+1}\log g^*_{t+1}|\mathcal{S}_t\right)$ or a Chernoff counterpart to it, as in Anderson, Hansen, and Sargent (2003).[11]

Our characterizations of worst case models have conditioned implicitly on the current

[10]See Johnsen and Donaldson (1985).

[11]Anderson, Hansen, and Sargent (2003) show a close connection between the market price of risk and a bound on the error probability for a statistical test for discriminating the approximating model from the worst case model.

period action. The implied distortion in the signal density is:

$$\int \phi_t(z^*, s^*)\tau(z^*, s^*|y_t, z, , a_t)\psi_t(z)q_t(z)d\lambda(z^*)d\lambda(z)$$

where ϕ_t is given by formula (14) and ψ_t is given by (17). When a Bellman-Isaacs condition is satisfied,[12] we can substitute for the control law and construct a conditional worst case conditional probability density for s_{t+1} as a function of the Markov state (y_t, q_t). The process $\{(y_{t+1}, q_{t+1}) : t \geq 0\}$ is Markov under the worst case distribution for the signal evolution. The density q_t remains a component of the state vector, even though it is not the worst case density for z_t.

8 A recursive multiple priors model

To attain a notion of dynamic consistency when the decision maker has multiple models, Epstein and Schneider (2003a) and Epstein and Schneider (2003b) advocate a formulation that, when translated into our setting, implies time varying values for θ_1 and θ_2. Epstein and Schneider advocate sequential constraints on sets of transition probabilities for signal distributions. To implement their proposal in our context, we can replace our fixed penalty parameters θ_1, θ_2 with two sequences of constraints on relative entropy.

In particular, suppose that

$$\varepsilon_t^1(m_{t+1}) \leq \kappa_t^1 \tag{32}$$

where κ_t^1 is a positive random variable in $\mathcal{X}_t$, and

$$\varepsilon_t^2(h_t) \leq \kappa_t^2 \tag{33}$$

where κ_t^2 is a positive random variable in $\mathcal{S}_t$. If these constraints bind, the worst-case probability distributions are again exponentially tilted. We can take θ_t^1 to be the $\mathcal{X}_t$-measurable Lagrange Multiplier on constraint (32), where $m_{t+1}^* \propto \exp\left(-\frac{W_{t+1}}{\theta_t^1}\right)$ and θ_t^1 solves $\varepsilon_t^1(m_{t+1}^*) = \kappa_t^1$. The counterpart to $\mathsf{R}_t^1(\mathsf{W}_{t+1})$ is

$$\mathsf{C}_t^1(\mathsf{W}_{t+1}) \doteq \frac{\mathsf{E}\left[\mathsf{W}_{t+1}\exp\left(-\frac{\mathsf{W}_{t+1}}{\theta_t^1}\right)|\mathcal{X}_t\right]}{\mathsf{E}\left[\exp\left(-\frac{\mathsf{W}_{t+1}}{\theta_t^1}\right)|\mathcal{X}_t\right]}.$$

Similarly, let θ_t^2 be the $\mathcal{S}_t$-measurable Lagrange multiplier on constraint (33), where $h_t^* \propto \exp\left(-\frac{\hat{W}_t}{\theta_t^2}\right)$, and θ_t^2 solves $\varepsilon_t^2(h_t^*) = \kappa_t^2$. The counterpart to $\mathsf{R}_t^2(\hat{\mathsf{W}}_t)$ is

$$\mathsf{C}_t^2(\hat{\mathsf{W}}_t) \doteq \frac{\mathsf{E}\left[\hat{\mathsf{W}}_t\exp\left(-\frac{\hat{\mathsf{W}}_t}{\theta_t^2}\right)|\mathcal{S}_t\right]}{\mathsf{E}\left[\exp\left(-\frac{\hat{\mathsf{W}}_t}{\theta_t^2}\right)|\mathcal{S}_t\right]}.$$

These constraint problems lead to natural counterparts to the operators T^1 and T^2.

[12]For example, see Hansen, Sargent, Turmuhambetova, and Williams (2004) or Hansen and Sargent (2004).

Constraint formulations provide a justification for making θ_1 and θ_2 state- or time-dependent. Values of θ_1 and θ_2 would coincide if the two constraints were replaced by a single entropy constraint $E\left[h_t\varepsilon_t^1(m_{t+1})|\mathcal{S}_t\right] + \varepsilon_t^2(h_t) \leq \kappa_t$, where κ_t is $\mathcal{S}_t$-measurable. Lin, Pan, and Wang (2004) and Maenhout (2004) give other reasons for making the robustness penalty parameters state dependent.[13] With such state dependence, it can still be useful to disentangle misspecifications of the state dynamics and the distribution of the hidden state given current information. Using separate values for θ_1 and θ_2 achieves that.

9 Risk sensitivity and compound lotteries

Jacobson (1973) pointed out a link between a concern about robustness, as represented in the first line of (11), and risk sensitivity, as conveyed in the second line of (11). That link has been exploited in the control theory literature, for example, by Whittle (1990). Our desire to separate the concern for misspecifying state dynamics from that for misspecifying the distribution of the state inspires two risk-sensitivity operators. Although our primary interest is in representing ways that the decision maker can respond to model misspecification, our two operators can also be interpreted in terms of enhanced risk aversion.[14]

9.1 Risk-sensitive interpretation of R_t^1

The R_t^1 operator has an alternative interpretation as a *risk-sensitive* adjustment to continuation values that expresses how a decision maker who has no concern about robustness prefers to adjust continuation values for their risk. The literature on risk-sensitive control uses adjustments of the same $\log E \exp$ form that emerge from an entropy penalty and a concern for robustness, as asserted in (11). There are risk adjustments that are more general than those of the $\log E \exp$ form associated with risk-sensitivity. In particular, we could follow Kreps and Porteus (1978) and Epstein and Zin (1989) in relaxing the assumption that a temporal compound lottery can be reduced to a simple lottery without regard to how the uncertainty is resolved, which would lead us to adjust continuation values by

$$\tilde{\mathsf{R}}_t^1(W_{t+1}) = \phi^{-1}\left(E\left[\phi(W_{t+1})|\mathcal{X}_t\right]\right)$$

for some concave increasing function ϕ. The risk-sensitive case is the special one in which ϕ is an exponential function. We focus on the special risk-sensitivity $\log E \exp$ adjustment because it allows us to use entropy to interpret the resulting adjustment as a way of inducing robust decision rules.

9.2 R_t^2 and the reduction of compound lotteries

While (17) shows that the operator R_t^2 assigns a worst-case probability distribution, another interpretation along the lines of Segal (1990), Klibanoff, Marinacci, and Mukerji (2003), and

[13]These authors consider problems without hidden states, but their motivation for state dependence would carry over to decision problems with hidden states.

[14]Using detection probabilities, Anderson, Hansen, and Sargent (2003) describe senses in which the risk-sensitivity and robustness interpretations are and are not observationally equivalent.

Ergin and Gul (2004) is available. This operator adjusts for state risk differently than does the usual Bayesian approach of model averaging. Specifically, we can regard the transformation R_t^2 as a version of what Klibanoff, Marinacci, and Mukerji (2003) call constant ambiguity aversion. More generally, we could use

$$\tilde{\mathsf{R}}_t^2(\hat{W}_t) = \psi^{-1} E\left[\psi(\hat{W}_t)|\mathcal{S}_t\right]$$

for some concave increasing function ψ. Again, we use the particular '$\log E \exp$' adjustment because of its explicit link to entropy-based robustness.

10 Linear quadratic problems

For a class of problems in which U is quadratic and the transition laws (1), (3), (2) are linear, this section describes how to use deterministic linear quadratic control problems to compute T^1, T^2, and $\mathsf{T}^2 \circ \mathsf{T}^1$. We consign details to appendix A. We begin with a remark that allows us to simplify the calculations by exploiting a type of certainty equivalence.

10.1 A useful form of certainty equivalence

We display the key idea in the following pair of problems that allow us easily to compute the T^1 operator. Problem 10.1 is a deterministic one-period control problem that recovers the objects needed to compute the T^1 operator defined in problem 10.2.

Problem 10.1. *Consider a quadratic value function $V(x) = -\frac{1}{2}x'\Omega x - \omega$, where Ω is a positive definite matrix. Consider the control problem*

$$\min_v V(x^*) + \frac{\theta}{2}|v|^2$$

subject to a linear transition function $x^* = Ax + Cv$. If θ is large enough that $I - \theta^{-1}C'\Omega C$ is positive definite, the problem is well posed and has solution

$$v = Kx \tag{34}$$
$$K = [\theta I - C'\Omega C]^{-1}C'\Omega A. \tag{35}$$

The following problem uses (34) and (35) to compute the T^1 operator:

Problem 10.2. *Consider the same value function $V(x) = -\frac{1}{2}x'\Omega x - \omega$ as in problem 10.1, but now let the transition law be*

$$x^* = Ax + Cw^*$$

where $w^* \sim \mathcal{N}(0, I)$. Consider the problem associated with the T^1 operator:

$$\min_{m^*} E[m^*V(x^*) + \theta m^* \log m^*].$$

The minimizer is

$$\begin{aligned} m^* &\propto \exp\left(\frac{-V(x^*)}{\theta}\right) \\ &= \exp\left[-\frac{1}{2}(w^*-v)'\Sigma^{-1}(w^*-v)+\frac{1}{2}w^*\cdot w^*-\frac{1}{2}\log\det\Sigma\right] \end{aligned}$$

where v is given by (34)-(35) from problem 10.1, $\Sigma=(I-\theta^{-1}C'\Omega C)^{-1}$, and the entropy of m^* is

$$Em^*\log m^*=\frac{1}{2}\Big[|v|^2+\text{trace}(\Sigma-I)-\log\det\Sigma\Big].$$

Thus, we can compute T^1 by solving the deterministic problem 10.1. We can also compute the T^2 and $\mathsf{T}^2\circ\mathsf{T}^1$ operators by solving appropriate deterministic control problems. In appendix A, we exploit certainty equivalence to compute these operators for the linear quadratic problem that we describe next.

10.2 The linear quadratic problem

This section specializes the general setup of section 2 by specifying a quadratic return function and a linear transition law. The return function or one period utility function is

$$U(x_t,a_t)=-\frac{1}{2}\begin{bmatrix}a_t' & x_t'\end{bmatrix}\begin{bmatrix}Q & P\\ P' & R\end{bmatrix}\begin{bmatrix}a_t\\ x_t\end{bmatrix}.$$

The transition laws are the following specializations of (1), (2), and (3):

$$\begin{aligned} y_{t+1} &= \Pi_s s_{t+1}+\Pi_y y_t+\Pi_a a_t \\ z_{t+1} &= A_{21}y_t+A_{22}z_t+B_2a_t+C_2w_{t+1} \\ s_{t+1} &= D_1y_t+D_2z_t+Ha_t+Gw_{t+1} \end{aligned} \tag{36}$$

where $w_{t+1}\sim\mathcal{N}(0,I)$ is an i.i.d. Gaussian vector process. Substituting from the evolution equation for the signal (36), we obtain:

$$y_{t+1}=(\Pi_sD_1+\Pi_y)y_t+\Pi_sD_2z_t+(\Pi_sH+\Pi_a)a_t+\Pi_sGw_{t+1},$$

which gives the y-rows in the following state-space system:

$$\begin{aligned} x_{t+1} &= Ax_t+Ba_t+Cw_{t+1} \\ s_{t+1} &= Dx_t+Ha_t+Gw_{t+1}, \end{aligned} \tag{37}$$

where $A_{11}\doteq\Pi_sD_1+\Pi_y$, $A_{12}\doteq\Pi_sD_2$, $B_1\doteq\Pi_sH+\Pi_a$ and $C_1\doteq\Pi_sG$.

Applying the Kalman filter to model (37) gives the following counterpart to (2), (4), $\kappa(s^*|y,z,a)$, and (6):

$$x^* = Ax+Ba+Cw^* \tag{38}$$
$$\check{z}^* = A_{21}y+A_{22}\check{z}+B_2a+K_2(\Delta)(s^*-\check{s}^*) \tag{39}$$
$$\Delta^* = A_{22}\Delta A_{22}'-K_2(\Delta)\left(A_{22}\Delta D_2'+C_2G'\right)'+C_2C_2' \tag{40}$$

where w^* is a standard normal random vector, $K_2(\Delta)$ is the Kalman gain

$$K_2(\Delta) = (A_{22}\Delta D_2{}' + C_2 G')(D_2 \Delta D_2{}' + GG')^{-1},$$

the innovation $s^* - \check{s}^* = D_2(z - \check{z}) + Gw^*$, and $\check{s}^*$ is the expectation of s^* conditioned on y_0 and the history of the signal. Equation (38) is the counterpart of (2), (4), while equations (39)-(40) form the counterpart to the law of motion for (sufficient statistics for) the posterior, $q^* = \pi_q(s^*, y, q, a)$. Under the approximating model, the hidden state z is a normally distributed random vector with mean $\check{z}$ and covariance matrix Δ. Equations (39) and describe the evolutions of the mean and covariance matrix of the hidden state, respectively.

10.3 Differences from situation under commitment

By *not* imposing distortions to $\check{z}$ and Δ on the right side of (39), the decision maker disregards prior distortions to the distribution of z. By way of contrast, in the commitment problem analyzed in Hansen and Sargent (2005), distortions to $\check{z}$ and Δ are present that reflect how past states and actions altered the worst case probability distribution for z.[15] Unlike the setting with commitment, in the present setup without commitment, $(\check{z}, \Delta)$ from the ordinary Kalman filter are state variables, just as in the standard linear quadratic control problem without a concern for robustness.

10.4 Perturbed models

The decision maker explores perturbations to the conditional distributions of w^* and z. Letting the altered distribution of w^* depend on the hidden state z allows for misspecification of the hidden state dynamics. Directly perturbing the conditional distribution of z is a convenient way to explore robustness to the filtered estimate of the hidden state associated with the approximating model. We perturb the distribution of w^* by applying the T^1 operator and the distribution of z by applying the T^2 operator. Section A.2 of appendix A exploits the certainty equivalence ideas conveyed in problem 10.1 to compute the $\mathsf{T}^2 \circ \mathsf{T}^1$ and T^1 operators for the recursions (21)-(22). Section A.3 describes how to compute the $\mathsf{T}^2 \circ \mathsf{T}^1$ operator of section 4 for formulating the game defined by the recursions (23) of section 5. The games in sections A.2 and A.3 allow $\theta_1 \neq \theta_2$. Section A.4 describes how to compute the composite operator (26) of section 6.2. The associated game requires that $\theta_1 = \theta_2$.

11 Concluding remarks

For a finite θ_1, the operator T^1 captures the decision maker's fear that the state and signal dynamics conditioned on both observed and hidden components of the state are misspecified. For a finite θ_2, the operator T^2 captures the decision maker's fear that the distribution of the hidden state conditioned on the history of signals is misspecified. Using different values of θ_1 and θ_2 in the operators T^1 and T^2 gives us the freedom to focus distrust on different aspects

[15]Hansen and Sargent (2005) also analyze a linear quadratic problem under commitment.

of the decision maker's model. That will be especially useful extensions of our framework to continuous time settings.

Specifications with $\theta_1 = \theta_2$ emerge when we follow Hansen and Sargent (2005) by adopting a timing protocol that requires the malevolent agent to commit to a worst case model $\{M_{t+1}\}$ once and for all at time 0. Hansen and Sargent (2005) give a recursive representation for the solution of the commitment problem in terms of R_t^1 and R_t^2 operators with a common but time-varying multiplier equal to $\frac{\theta}{\beta^t}$. The presence of β^t causes the decision maker's concerns about misspecification to vanish for tail events. Only for the undiscounted case does the zero-sum two player game with commitment in Hansen and Sargent (2005) give identical outcomes to the recursive games in this paper. As noted in section 6.6, when $\beta < 1$, the gap between the outcomes with and without commitment is the source of time-inconsistency of the worst case beliefs about the hidden state.

Much of the control theory literature (e.g., Whittle (1990) and Basar and Bernhard (1995)) uses the commitment timing protocol. Hansen and Sargent (2005) show how to represent parts of that literature in terms of our formulation of model perturbations as martingales.

A Computations for LQ problems

A.1 Three games

We use the certainty equivalence insight from subsection 10.1 to solve three games. The key step in each is to formulate an appropriate linear quadratic discounted dynamic programming problem. Game I enables us to compute the $\mathsf{T}^2 \circ \mathsf{T}^1$ and the T^1 operators required by recursions (21)-(22). Game II formulates a linear regulator that we use to compute the recursions in formulation (23). Game III formulates a recursion for the operator (26) that is pertinent when $\theta_1 = \theta_2$.

A.2 Game I

This subsection shows how to apply the certainty equivalent insight from section 10.1 to compute the recursions (21)-(22) (i.e., "maximize after applying $\mathsf{T}^2 \circ \mathsf{T}^1$, but update by applying T^1") for the linear quadratic case. In game I, a decision maker chooses a after first applying $\mathsf{T}^2 \circ \mathsf{T}^1$ to the sum of the current return function and a discounted continuation value. This makes a depend on y and the estimate of the hidden state $\check{z}$, but not on z. However, by updating the value function using T^1 only, we make the continuation value function depend on the hidden state z. We adopt the convention that we discount the continuation value function and then add to it the current return function and the undiscounted penalties on the two entropies.

Rewrite evolution equation (38) - (39) as

$$\begin{aligned}
\begin{bmatrix} y^* \\ z^* \\ \check{z}^* \end{bmatrix} &= \begin{bmatrix} A_{11} & A_{12} & 0 \\ A_{21} & A_{22} & 0 \\ A_{21} & K_2(\Delta)D_2 & A_{22} - K_2(\Delta)D_2 \end{bmatrix} \begin{bmatrix} y \\ z \\ \check{z} \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \\ B_2 \end{bmatrix} a + \begin{bmatrix} C_1 \\ C_2 \\ K_2(\Delta)G \end{bmatrix} w^* \\
&= \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} y \\ \check{z} \end{bmatrix} + \begin{bmatrix} B_1 & A_{12} \\ B_2 & A_{22} \\ B_2 & K_2(\Delta)D_2 \end{bmatrix} \begin{bmatrix} a \\ z - \check{z} \end{bmatrix} + \begin{bmatrix} C_1 \\ C_2 \\ K_2(\Delta)G \end{bmatrix} w^*. \qquad (41)
\end{aligned}$$

Under the approximating model, w^* is a multivariate standard normal random vector and $z - \check{z}$ is distributed as a normal random vector with mean zero and covariance matrix Δ.

A.2.1 Computing $\mathsf{T}^1 \circ \mathsf{T}^2$

The logic expressed in (11) and (15) that define R_t^1 and R_t^2 shows that application of $\mathsf{T}^2 \circ \mathsf{T}^1$ to a function amounts to minimizing another function with respect to the distributions of z and w^*. We shall exploit this logic and calculate $\mathsf{T}^2 \circ \mathsf{T}^1$ by solving the corresponding minimization problem. In the present linear-quadratic-Gaussian case, we can exploit the certainty equivalence property from section 10.1 and minimize first over the conditional means of these two distributions, then construct the minimizing conditional covariances later, thereby exploiting the idea in problem 10.2. Because a Bellman-Isaacs condition is satisfied, the linear-quadratic-Gaussian structure allows us simultaneously to perform the maximization over a and the minimization over the distorted means of z and w^* associated with the $\mathsf{T}^2 \circ \mathsf{T}^1$ operator. We do this by forming a zero-sum two-player game that simultaneously chooses the decision a, a distortion u to the mean of $z - \check{z}$, and a distortion $\tilde{v}$ to a conditional mean of w^*, all as functions of $y, \check{z}$. Here $\tilde{v}$ can be interpreted as the mean of v conditioned on $y, \check{z}$. (In section A.2.2, we shall compute a vector v that is the distorted mean of w^* conditioned on $y, \check{z}, z$.)[16]

Thus, we consider the transition equation:

$$\begin{bmatrix} y^* \\ z^* \\ \check{z}^* \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} y \\ \check{z} \end{bmatrix} + \begin{bmatrix} B_1 & A_{12} \\ B_2 & A_{22} \\ B_2 & K_2(\Delta) D_2 \end{bmatrix} \begin{bmatrix} a \\ u \end{bmatrix} + \begin{bmatrix} C_1 \\ C_2 \\ K_2(\Delta) G \end{bmatrix} \tilde{v}$$

Write the single period objective as:

$$-\frac{1}{2} \begin{bmatrix} a' & y' & z' \end{bmatrix} \begin{bmatrix} Q & P_1 & P_2 \\ P_1' & R_{11} & R_{12} \\ P_2' & R_{21} & R_{22} \end{bmatrix} \begin{bmatrix} a \\ y \\ z \end{bmatrix} + \frac{\theta_1}{2} |\tilde{v}|^2 + \frac{\theta_2}{2} u' \Delta^{-1} u = -\frac{1}{2} \begin{bmatrix} a' & u' & \tilde{v}' & y' & \check{z}' \end{bmatrix} \Pi(\Delta) \begin{bmatrix} a \\ u \\ \tilde{v} \\ y \\ \check{z} \end{bmatrix}$$

where

$$\Pi(\Delta) = \begin{bmatrix} Q & P_2 & 0 & P_1 & P_2 \\ P_2' & R_{22} - \theta_2 \Delta^{-1} & 0 & R_{21} & R_{22} \\ 0 & 0 & -\theta_1 I & 0 & 0 \\ P_1' & R_{12} & 0 & R_{11} & R_{12} \\ P_2' & R_{22} & 0 & R_{21} & R_{22} \end{bmatrix}. \tag{42}$$

Construct a composite action vector:

$$\tilde{a} = \begin{bmatrix} a \\ u \\ \tilde{v} \end{bmatrix} \tag{43}$$

and composite state vector

$$\tilde{x} = \begin{bmatrix} y \\ \check{z} \end{bmatrix}. \tag{44}$$

Express (41) as

$$\begin{bmatrix} y^* \\ z^* \\ \check{z}^* \end{bmatrix} = \tilde{A} \tilde{x} + \tilde{B}(\Delta) \tilde{a},$$

and the single period objective as

$$-\frac{1}{2} \begin{bmatrix} \tilde{a}' & \tilde{x}' \end{bmatrix} \begin{bmatrix} \Pi_{11}(\Delta) & \Pi_{12} \\ \Pi_{21} & \Pi_{22} \end{bmatrix} \begin{bmatrix} \tilde{a} \\ \tilde{x} \end{bmatrix},$$

and write the discounted next period value function as

$$\beta V^*(y^*, z^*, \check{z}^*, \Delta^*) = -\frac{\beta}{2} \begin{bmatrix} y^{*\prime} & z^{*\prime} & \check{z}^{*\prime} \end{bmatrix} \Omega^*(\Delta^*) \begin{bmatrix} y^* \\ z^* \\ \check{z}^* \end{bmatrix} - \beta \omega(\Delta^*).$$

[16]Note that T^1 makes v depend on $y, z, \check{z}$, and that application of T^2 then conditions down to $y, \check{z}$, in effect recovering the mean of v conditional on $(y, \check{z})$.

Then pose the problem[17]

$$\max_{a}\min_{u,\tilde{v}}\Big\{-\frac{1}{2}\begin{bmatrix}\tilde{a}' & \tilde{x}'\end{bmatrix}\begin{bmatrix}\Pi_{11}(\Delta) & \Pi_{12}\\ \Pi_{21} & \Pi_{22}\end{bmatrix}\begin{bmatrix}\tilde{a}\\ \tilde{x}\end{bmatrix}+\beta V^*(y^*,z^*,\check{z}^*,\Delta^*)\Big\}. \tag{45}$$

The composite decision rule is

$$\tilde{a}=-\left[\Pi_{11}(\Delta)+\beta\tilde{B}(\Delta)'\Omega^*(\Delta^*)\tilde{B}(\Delta)\right]^{-1}\left[\Pi_{12}+\beta\tilde{B}(\Delta)'\Omega^*(\Delta^*)\tilde{A}\right]\tilde{x}.$$

Using the law of motion from the Kalman filter

$$\Delta^*=A_{22}\Delta A_{22}'-K(\Delta)\left(A_{22}\Delta D_2'+C_2G'\right)'+C_2C_2'$$

to express Δ^* in terms of Δ, the composite decision rule can be expressed as

$$\begin{bmatrix}a\\ u\\ \tilde{v}\end{bmatrix}\doteq-\tilde{F}(\Delta)\begin{bmatrix}y\\ z\\ \check{z}\end{bmatrix}=-\begin{bmatrix}\tilde{F}_{11}(\Delta) & 0 & \tilde{F}_{13}(\Delta)\\ \tilde{F}_{21}(\Delta) & 0 & \tilde{F}_{23}(\Delta)\\ \tilde{F}_{31}(\Delta) & 0 & \tilde{F}_{33}(\Delta)\end{bmatrix}\begin{bmatrix}y\\ z\\ \check{z}\end{bmatrix}, \tag{46}$$

which looks like the decision rule for an optimal linear regulator problem. The robust control law for the action is given by the first block in (46). In the second line, we have added z, which at this stage is a superfluous component of the state vector, so that the corresponding columns of $\tilde{F}(\Delta)$ are identically zero; $\tilde{a}$ is by construction a function of $(y,\check{z})$. This superfluous state variable will be useful in section A.2.2 when we compute a continuation value function that depends on $(y,z,\check{z})$.

To make the extremization in (45) well posed, we require that θ_1,θ_2 be large enough to satisfy the 'no-breakdown' condition that

$$\begin{bmatrix}\theta_2\Delta^{-1}-R_{22} & 0\\ 0 & \theta_1 I\end{bmatrix}-\beta\begin{bmatrix}A_{12}' & A_{22}' & D_2'K_2(\Delta)'\\ C_1' & C_2' & G'K_2(\Delta)'\end{bmatrix}\Omega^*(\Delta^*)\begin{bmatrix}A_{12} & C_1\\ A_{22} & C_2\\ K_2(\Delta)D_2 & K_2(\Delta)G\end{bmatrix}$$

is positive definite. Otherwise, the parameter pair (θ_1,θ_2) is not admissible. This is a bivariate counterpart to a check for a no-breakdown condition that occurs in robust control theory. When the no-breakdown condition is violated, the minimizing agent can make the objective equal to $-\infty$.

A.2.2 T^1 and the worst case $E[w^*|y,z,\check{z}]$

It remains for us to compute the distortion to the mean of w^* conditional on $y,z,\check{z}$ that emerges from applying the T^1 operator to a continuation value. The T^1 operator allows a minimizing agent to exploit his information advantage over the maximizing agent by letting the mean distortion in w^* depend on z, the part of the state that is hidden from the maximizing agent.

[17] Note here how we discount the continuation value function, then add the current return and the penalized entropies.

Taking the control law for a computed in (46) as given, we can compute the mean v of the worst case w^* conditional on $y, z, \check{z}$ by using the evolution equation (41):

$$\begin{aligned}\begin{bmatrix} y^* \\ z^* \\ \check{z}^* \end{bmatrix} &= \begin{bmatrix} A_{11} & A_{12} & 0 \\ A_{21} & A_{22} & 0 \\ A_{21} & K_2(\Delta)D_2 & A_{22} - K_2(\Delta)D_2 \end{bmatrix} \begin{bmatrix} y \\ z \\ \check{z} \end{bmatrix} - \begin{bmatrix} B_1 \\ B_2 \\ B_2 \end{bmatrix} \tilde{F}_1(\Delta) \begin{bmatrix} y \\ z \\ \check{z} \end{bmatrix} + \begin{bmatrix} C_1 \\ C_2 \\ K_2(\Delta)G \end{bmatrix} v \\ &= \bar{A}(\Delta) \begin{bmatrix} y \\ z \\ \check{z} \end{bmatrix} + \bar{C}(\Delta)v.\end{aligned}$$

After substituting the decision rule for a from (46), we can write the objective as

$$-\frac{1}{2} \begin{bmatrix} a' & y' & z' \end{bmatrix} \begin{bmatrix} Q & P_1 & P_2 \\ P_1{}' & R_{11} & R_{12} \\ P_2{}' & R_{21} & R_{22} \end{bmatrix} \begin{bmatrix} a \\ y \\ z \end{bmatrix} + \frac{\theta_1}{2}|v|^2 = -\frac{1}{2} \begin{bmatrix} y' & z' & \check{z}' \end{bmatrix} \bar{\Pi}(\Delta) \begin{bmatrix} y \\ z \\ \check{z} \end{bmatrix} + \frac{\theta_1}{2}|v|^2$$

where

$$\bar{\Pi}(\Delta) \doteq \begin{bmatrix} -\tilde{F}_1(\Delta) \\ I \quad 0 \quad 0 \\ 0 \quad I \quad 0 \\ 0 \quad 0 \quad I \end{bmatrix}' \begin{bmatrix} Q & P_1 & P_2 & 0 \\ P_1{}' & R_{11} & R_{12} & 0 \\ P_2{}' & R_{21} & R_{22} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} -\tilde{F}_1(\Delta) \\ I \quad 0 \quad 0 \\ 0 \quad I \quad 0 \\ 0 \quad 0 \quad I \end{bmatrix}.$$

Provided that

$$\theta_1 I - \beta \bar{C}(\Delta)' \Omega^*(\Delta^*) \bar{C}(\Delta) \tag{47}$$

is positive definite, the control law for v is[18]

$$v = \beta \left[\theta_1 I - \beta \bar{C}(\Delta)' \Omega^*(\Delta^*) \bar{C}(\Delta) \right]^{-1} \bar{C}(\Delta)' \Omega^*(\Delta^*) \bar{A} \begin{bmatrix} y \\ z \\ \check{z} \end{bmatrix}, \tag{48}$$

which, by using (40) to express Δ^* as a function of Δ, we can express as

$$v = \bar{F}(\Delta) \begin{bmatrix} y \\ z \\ \check{z} \end{bmatrix}.$$

The updated value function is

$$\begin{aligned}\Omega(\Delta) &= \bar{\Pi}(\Delta) + \beta \bar{A}' \Omega^*(\Delta^*) \bar{A} \\ &\quad + \beta^2 \bar{A}' \Omega^*(\Delta^*) \bar{C}(\Delta) \left[\theta_1 I - \beta \bar{C}(\Delta)' \Omega^*(\Delta^*) \bar{C}(\Delta) \right]^{-1} \bar{C}(\Delta)' \Omega^*(\Delta^*) \bar{A}.\end{aligned} \tag{49}$$

At first sight, these recursions seem difficult because they call for updating the matrix valued functions Ω for *all* hypothetical values of the definite matrix Δ. Fortunately, it suffices to perform these calculations only for a sequence of Δ's calculated over the horizon of interest, which is easy. Given a sequence of Δ's starting from an initial condition, the Ω's for the value functions can be computed starting from a terminal value using backward induction. In particular, we can first compute a sequence of matrices Δ using forward induction on (40), then compute a corresponding $\Omega(\Delta)$ sequence using backward induction on (49). Both forward and backward recursions are Riccati equations.

[18]If the matrix defined in (47) is not positive definite, then θ_1 is below the break-down point.

A.2.3 Worst case shock distribution

The worst case distribution for w^* conditioned on $(y, z, \check{z})$ is normal with mean v given by (48) and covariance

$$\Sigma(\Delta) \doteq \left[I - \frac{\beta}{\theta_1}\bar{C}'\Omega^*(\Delta^*)\bar{C}\right]^{-1}.$$

A.2.4 Worst case hidden state distribution

The worst case mean of z conditional on $(y, \check{z})$ is

$$u = -\tilde{F}_2(\Delta)\begin{bmatrix} y \\ z \\ \check{z} \end{bmatrix},$$

(recall that $\tilde{F}_2$ contains zeros in the columns that multiply z), and its covariance matrix is:

$$\Gamma(\Delta) \doteq \left(\Delta^{-1} - \frac{1}{\theta_2}R_{22} - \frac{1}{\theta_2}\begin{bmatrix} 0 & I & 0 \end{bmatrix}\Omega(\Delta)\begin{bmatrix} 0 \\ I \\ 0 \end{bmatrix}\right)^{-1},$$

provided that this matrix is positive definite. Otherwise, θ_2 is below its breakdown point.

A.2.5 Consistency check

The third row of (46) computes the mean $\tilde{v}$ of w^*, conditional on the information set available to the maximizing agent, namely, $(y, \check{z})$, but not z. In formula (48), we computed the mean v of w^* conditional on the information set of the minimizing agent, namely, $(y, z, \check{z})$. A certainty equivalence result asserts that $\tilde{v}$ is the expectation of v conditioned on $(y, \check{z})$. This gives us the following consistency check.

One formula for $\tilde{v}$ is computed by using the control law of v and substituting for the distorted expectation for z:

$$\tilde{v} = \bar{F}(\Delta)\begin{bmatrix} y \\ \check{z} - \tilde{F}_{21}(\Delta)y - \tilde{F}_{23}(\Delta)\check{z} \\ \check{z} \end{bmatrix} = \bar{F}(\Delta)\begin{bmatrix} I & 0 \\ -\tilde{F}_{21}(\Delta) & I - \tilde{F}_{23}(\Delta) \\ 0 & I \end{bmatrix}\begin{bmatrix} y \\ \check{z} \end{bmatrix}.$$

Using certainty equivalence, we computed $\tilde{v} = -\tilde{F}_3\begin{bmatrix} y \\ \check{z} \end{bmatrix}$. Taken together, we have the restriction

$$-\begin{bmatrix} \tilde{F}_{31}(\Delta) & \tilde{F}_{33}(\Delta) \end{bmatrix} = \bar{F}(\Delta)\begin{bmatrix} I & 0 \\ -\tilde{F}_{21}(\Delta) & I - \tilde{F}_{23}(\Delta) \\ 0 & I \end{bmatrix}.$$

A.2.6 Worst case signal distribution

In this section, we recursively construct the distribution of signals under the distorted probability distribution. Recall the signal evolution:

$$s^* = Dx + Ha + Gw.$$

Under the approximating model, the signal next period is normal with mean

$$\check{s}^* = D_1 y + D_2 \check{z} + Ha$$

and covariance matrix

$$\check{\Upsilon} = D_2 \Delta D_2{}' + GG'.$$

The distorted mean of the signal conditioned on the signal history is:

$$\bar{s}^* = D_1 y + D_2 \check{z} + (D_2 u + G\tilde{v}) + Ha$$

which by virtue of the second and third blocks of rows of (46) can be written

$$\bar{s}^* = \bar{D}_1(\Delta) y + \bar{D}_2(\Delta)\check{z} + Ha \tag{50}$$

where

$$\begin{aligned}
\bar{D}_1(\Delta) &\doteq D_1 - D_2 \tilde{F}_{21}(\Delta) - G\tilde{F}_{31}(\Delta) \\
\bar{D}_2(\Delta) &\doteq D_2 - D_2 \tilde{F}_{23}(\Delta) - G\tilde{F}_{33}(\Delta).
\end{aligned}$$

The distorted covariance matrix is:

$$\bar{\Upsilon} = D_2 \Gamma(\Delta) D_2{}' + G\Sigma(\Delta)G'.$$

The relative entropy of this distortion conditioned on the reduced information set of the signal history is

$$\frac{1}{2}\left[(\bar{s}^* - \check{s}^*)'\check{\Upsilon}^{-1}(\bar{s}^* - \check{s}^*) + \mathrm{trace}(\check{\Upsilon}^{-1}\bar{\Upsilon} - I) - \mathrm{logdet}\bar{\Upsilon} + \mathrm{logdet}\check{\Upsilon}\right]. \tag{51}$$

To construct the distorted dynamics for y^*, start from the formula for y^* from the first block in (36), namely, $y^* = \Pi_s s^* + \Pi_y y + \Pi_a a$. Substituting for the robust decision rule for a from the first block of row of (46) and replacing s^* with with $\bar{s}^* + (s^* - \bar{s}^*)$ from (50) gives

$$y^* = [\Pi_y y + \Pi_s \bar{D}_1(\Delta) - (\Pi_s H + \Pi_a)\tilde{F}_{11}(\Delta)]y + [\Pi_s \bar{D}_2(\Delta) - (\Pi_s H + \Pi_a)\tilde{F}_{13}(\Delta)]\check{z} + \Pi_s(s^* - \bar{s}^*). \tag{52}$$

To complete a recursive representation for y^* under the worst case distribution, we need a formula for updating $\check{z}^*$ under the worst case distribution. Recall the formula for $\check{z}^*$ under the approximating model from the Kalman filter (39) or (41):

$$\check{z}^* = [A_{21} - B_2 \tilde{F}_{11}(\Delta)]y + [A_{22} - B_2 \tilde{F}_{13}(\Delta)]\check{z} + K_2(\Delta)(s^* - D_1 y - D_2 \check{z} - Ha)$$

or

$$\check{z}^* = [A_{21} - B_2 \tilde{F}_{11}(\Delta)]y + [A_{22} - B_2 \tilde{F}_{13}(\Delta)]\check{z} + K_2(\Delta)(s^* - \check{s}^*).$$

Using the identity

$$\begin{aligned} s^* - \check{s}^* &= (s^* - \bar{s}^*) + (\bar{s}^* - \check{s}^*) \\ &= (s^* - \bar{s}^*) + \big([\bar{D}_1(\Delta) - D_1]y + [\bar{D}_2(\Delta) - D_2]\check{z}\big) \end{aligned}$$

in the above equation gives:

$$\begin{aligned} \check{z}^* &= \Big(A_{21} - B_2\tilde{F}_{11}(\Delta) + K_2(\Delta)[\bar{D}_1(\Delta) - D_1]\Big)\, y \\ &+ \Big(A_{22} - B_2\tilde{F}_{13}(\Delta) + K_2(\Delta)[\bar{D}_2(\Delta) - D_2]\Big)\, \check{z} + K_2(\Delta)(s^* - \bar{s}^*). \end{aligned} \tag{53}$$

Taken together, (52) and (53) show how to construct $\check{z}^*$ from the signal history under the distorted law of motion. The innovation $s^* - \bar{s}^*$ under the distorted model is normal with mean zero and covariance matrix $\bar{\Upsilon}$.

A.3 Game II

We now turn to the linear quadratic version of a game associated with the recursion (23) described in section 5.2, in which we update the value function using $\mathsf{T}^2 \circ \mathsf{T}^1$. We exploit our certainty equivalence insights from section 10.1. Like Game I, this game allows $\theta_1 \neq \theta_2$. Here we do not need to keep track of the evolution of z^*. Instead it suffices to focus only on the two equation system:

$$\begin{bmatrix} y^* \\ \check{z}^* \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} y \\ \check{z} \end{bmatrix} + \begin{bmatrix} A_{12} \\ K_2(\Delta)D_2 \end{bmatrix} (z - \check{z}) + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} a + \begin{bmatrix} C_1 \\ K_2(\Delta)G \end{bmatrix} w^* \tag{54}$$

As in Game I, we need to choose the mean distortion u for $z - \check{z}$, and the mean distortion v for w, where both means distortions are conditioned on $(y, \check{z})$.

A.3.1 Computing a, u, and $\tilde{v}$

We apply the same argument as for Game I, but to a smaller state vector. Thus, we work with the evolution equation

$$\begin{bmatrix} y^* \\ \check{z}^* \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} y \\ \check{z} \end{bmatrix} + \begin{bmatrix} A_{12} \\ K_2(\Delta)D_2 \end{bmatrix} u + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} a + \begin{bmatrix} C_1 \\ K_2(\Delta)G_2 \end{bmatrix} \tilde{v}$$

or

$$\tilde{x}^* = \tilde{A}x + \tilde{B}(\Delta)\tilde{a},$$

where $\tilde{x}$ and $\tilde{a}$ are defined in (43) and (44) and $\tilde{x}^*$ is the next period's value of $\tilde{x}$. The matrices A and $\tilde{B}$ differ from those in Game I because z^* is not included in $\tilde{x}^*$.

Partition blocks of the matrix $\Pi(\Delta)$ defined in (42) as $\begin{bmatrix} \Pi_{11} & \Pi_{12} \\ \Pi_{21} & \Pi_{22} \end{bmatrix}$ conformably with $\tilde{a}, \tilde{x}$, so that the $(1,1)$ block pertains to $\tilde{a}$, the $(2,2)$ block to $\tilde{x}$, and so on. Write the discounted next period value function as

$$\beta V(\tilde{x}^*) = -\frac{\beta}{2}(\tilde{x}^*)'\Omega^*(\Delta^*)\tilde{x}^* - \beta\omega^*(\Delta^*).$$

Then the composite robust control is:

$$\begin{aligned}
\tilde{a} &= -\left[\Pi_{11}(\Delta) + \beta\tilde{B}(\Delta)'\Omega^*(\Delta^*)\tilde{B}(\Delta)\right]^{-1}\left[\Pi_{12} + \beta\tilde{B}(\Delta)'\Omega^*(\Delta^*)\tilde{A}\right]\tilde{x} \\
&\doteq -\begin{bmatrix}\tilde{F}_1(\Delta)\\ \tilde{F}_2(\Delta)\\ \tilde{F}_3(\Delta)\end{bmatrix}\tilde{x} \qquad (55)
\end{aligned}$$

where $-\tilde{F}_1(\Delta)\tilde{x}$ is the control law for a, $-\tilde{F}_2(\Delta)\tilde{x}$ is the control law for the mean u of the distorted distribution for $z - \check{z}$, and $-\tilde{F}_3(\Delta)\tilde{x}$ is the control law for $\tilde{v}$, the mean of the distorted distribution for w^* conditional on $(y, \check{z})$.

For the extremization problem to be well posed, we require that (θ_1, θ_2) be large enough that

$$\begin{bmatrix}\theta_2\Delta^{-1} - R_{22} & 0\\ 0 & \theta_1 I\end{bmatrix} - \beta\begin{bmatrix}A_{12}' & D_2'K_2(\Delta)'\\ C_1' & G'K_2(\Delta)'\end{bmatrix}\Omega(\Delta^*)\begin{bmatrix}A_{12} & C_1\\ K_2(\Delta)D_2 & K_2(\Delta)G\end{bmatrix} \qquad (56)$$

is positive definite.

The value function recursion is the Riccati equation:

$$\begin{aligned}
\Omega(\Delta) &= \Pi_{22} + \beta\tilde{A}(\Delta)'\Omega^*(\Delta^*)\tilde{A}(\Delta) - \left[\Pi_{12} + \beta\tilde{B}(\Delta)'\Omega^*(\Delta^*)\tilde{A}\right]' \\
&\quad \left[\Pi_{11}(\Delta) + \beta\tilde{B}(\Delta)'\Omega^*(\Delta^*)\tilde{B}(\Delta)\right]^{-1}\left[\Pi_{12} + \beta\tilde{B}(\Delta)'\Omega^*(\Delta^*)\tilde{A}\right].
\end{aligned}$$

This recursion computes a matrix in the quadratic form that emerges from applying the composite $\mathsf{T}^2 \circ \mathsf{T}^1$ operator.

A.3.2 Worst case distribution for w^* conditional on $(y, \check{z}, z)$

We now compute the mean v of the distorted distribution for w^* that emerges from applying the T^1 operator alone to the continuation value. The mean distortion v depends on the hidden state z, as well as on $(y, \check{z})$. To prepare the minimization problem that we use to compute T^1, first impose the control law for a in evolution equation (54):

$$\begin{aligned}
\begin{bmatrix}y^*\\ \check{z}^*\end{bmatrix} &= \tilde{A}\begin{bmatrix}y\\ \check{z}\end{bmatrix} - \begin{bmatrix}B_1\\ B_2\end{bmatrix}\tilde{F}_1(\Delta)\begin{bmatrix}y\\ \check{z}\end{bmatrix} + \begin{bmatrix}A_{12}\\ K_2(\Delta)D_2\end{bmatrix}(z - \check{z}) + \begin{bmatrix}C_1\\ K_2(\Delta)G\end{bmatrix}w^* \\
&= \bar{A}(\Delta)\begin{bmatrix}y\\ \check{z}\end{bmatrix} + \bar{H}(\Delta)(z - \check{z}) + \bar{C}(\Delta)w^*. \qquad (57)
\end{aligned}$$

The following certainty-equivalent problem recovers the feedback law for v associated with T^1:

$$\min_v -\frac{\beta}{2}\begin{bmatrix}y^{*\prime} & \check{z}'\end{bmatrix}\Omega^*(\Delta^*)\begin{bmatrix}y^*\\ \check{z}^*\end{bmatrix} + \frac{\theta_1}{2}v'v$$

where the minimization is subject to (57) with v replacing w^*. The minimizing v, which is the worst case mean of w^* conditional on $(y, \check{z}, z)$, is

$$v = \beta\left[\theta_1 I - \beta\bar{C}(\Delta)'\Omega^*(\Delta^*)\bar{C}(\Delta)\right]^{-1}\bar{C}(\Delta)'\Omega^*(\Delta^*)\left[\bar{A}\begin{bmatrix}y\\ \check{z}\end{bmatrix} + \bar{H}(\Delta)(z - \check{z})\right]$$

$$
\begin{aligned}
&= -\bar{F}_1(\Delta)(z-\check{z}) - \bar{F}_2(\Delta)\begin{bmatrix} y \\ \check{z} \end{bmatrix} \\
&= -\bar{F}(\Delta)\begin{bmatrix} z-\check{z} \\ y \\ \check{z} \end{bmatrix}.
\end{aligned}
$$

Conditional on $(y, \check{z}, z)$, the covariance matrix of the worst case w^* is

$$
\Sigma(\Delta) = \left[I - \frac{\beta}{\theta_1}\bar{C}(\Delta)'\Omega^*(\Delta^*)\bar{C}(\Delta)\right]^{-1}, \tag{58}
$$

which is positive definite whenever the breakdown condition (56) is met.

Next, we want to compute the matrix $\bar{\Omega}(\Delta)$ in the quadratic form in $\begin{bmatrix}(z-\check{z})' & y' & \check{z}'\end{bmatrix}$ that emerges from applying the T^1 operator. First, adjust the objective for the choice of v by constructing a matrix $\bar{\Pi}(\Delta)$, with row and column dimension both equal to the dimension of $\begin{bmatrix}(z-\check{z})' & y' & \check{z}'\end{bmatrix}$, that we now redefine as:[19]

$$
\bar{\Pi}(\Delta) = \begin{bmatrix} 0 & -\tilde{F}_1(\Delta) \\ I & 0 \quad 0 \\ 0 & I \quad 0 \\ 0 & 0 \quad I \end{bmatrix}' \begin{bmatrix} Q & P_2 & P_1 & P_2 \\ P_2' & R_{22} - \theta_2\Delta^{-1} & R_{21} & R_{22} \\ P_1' & R_{21} & R_{11} & R_{12} \\ P_2' & R_{22} & R_{21} & R_{22} \end{bmatrix} \begin{bmatrix} 0 & -\tilde{F}_1(\Delta) \\ I & 0 \quad 0 \\ 0 & I \quad 0 \\ 0 & 0 \quad I \end{bmatrix}.
$$

The matrix in the quadratic form in $\begin{bmatrix}(z-\check{z})' & y' & \check{z}'\end{bmatrix}$ for the minimized objective function that emerges from applying the T^1 operator is:

$$
\begin{aligned}
\bar{\Omega}(\Delta) \quad &= \quad \bar{\Pi}(\Delta) + \beta\begin{bmatrix} \bar{H}(\Delta)' \\ \bar{A}(\Delta)' \end{bmatrix}\Omega^*(\Delta^*)\begin{bmatrix} \bar{H}(\Delta) & \bar{A}(\Delta) \end{bmatrix} + \\
&\beta^2\begin{bmatrix} \bar{H}(\Delta)' \\ \bar{A}(\Delta)' \end{bmatrix}\Omega^*(\Delta^*)\bar{C}(\Delta)\left[\theta_1 I - \beta\bar{C}(\Omega)'\Omega^*(\Delta^*)\bar{C}(\Delta)\right]^{-1}\bar{C}(\Delta)'\Omega^*(\Delta^*)\begin{bmatrix} \bar{H}(\Delta) & \bar{A}(\Delta) \end{bmatrix}.
\end{aligned}
$$

A.3.3 Worst case distribution for $z - \check{z}$: $\mathcal{N}(u, \Gamma(\Delta))$

Knowing $\bar{\Omega}(\Delta)$ allows us to deduce the worst case distribution for $z - \check{z}$ conditional on $(y, \check{z})$ in another way, thereby establishing a useful cross check on formula (55). Use the partition:

$$
\bar{\Omega}(\Delta) = \begin{bmatrix} \bar{\Omega}_{11}(\Delta) & \bar{\Omega}_{12}(\Delta) \\ \bar{\Omega}_{21}(\Delta) & \bar{\Omega}_{22}(\Delta) \end{bmatrix}
$$

where $\bar{\Omega}_{11}(\Delta)$ has the same dimension as $z - \check{z}$ and $\bar{\Omega}_{22}(\Delta)$ has the same dimension as $\begin{bmatrix} y \\ \check{z} \end{bmatrix}$. The covariance matrix of $z - \check{z}$ is

$$
\Gamma(\Delta) = -\left[\frac{1}{\theta_2}\bar{\Omega}_{11}(\Delta)\right]^{-1}, \tag{59}
$$

[19]Note that we are recycling and changing notation from section A.2.

which is positive definite when (θ_1, θ_2) satisfies the no-breakdown restriction (56). The mean of the distorted distribution of $z - \check{z}$ is

$$u = -\left[\bar{\Omega}_{11}(\Delta)\right]^{-1} \bar{\Omega}_{12}(\Delta) \begin{bmatrix} y \\ \check{z} \end{bmatrix}.$$

Computing u at this stage serves as a consistency check because it was already computed; it must be true that

$$\tilde{F}_2(\Delta) = \left[\bar{\Omega}_{11}(\Delta)\right]^{-1} \bar{\Omega}_{12}(\Delta).$$

Given this choice of u, a second consistency check compares the formula for $\tilde{v}$ to the formulas for v and u; $\tilde{v}$ is a distorted expectation of v conditioned on y and $\check{z}$. Thus,

$$\tilde{F}_3(\Delta) = \bar{F}(\Delta) \begin{bmatrix} -\tilde{F}_2(\Delta) \\ I \end{bmatrix}.$$

A.3.4 Worst case signal distribution

The mean of the distorted signal distribution given the signal history for Game II is

$$\bar{s}^* = \left[D - D_2\tilde{F}_2(\Delta) - G\tilde{F}_3(\Delta)\right] \tilde{x},$$

and the distorted covariance matrix is:

$$\bar{\Upsilon} = D_2\Gamma(\Delta)D_2{}' + G\Sigma(\Delta)G'$$

with the Game II versions of $\Sigma(\Delta)$ and $\Gamma(\Delta)$ given by (58) and (59), respectively. The reduced information measure of entropy is given again by formula (51). The worst case evolution for y^* and $\check{z}^*$ expressed in terms of $s^* - \bar{s}^*$ is constructed as in Game I in formulas (52) and (53), but using the Game II control law $\tilde{F}_1$ for a.

A.4 Game III

Game III applies our certainty equivalence insight from section 10.1 to compute iterations on (26). This game assumes that $\theta_1 = \theta_2$, presumes that the period objective function does not depend on the hidden state, and works entirely with the reduced information set $y, \check{z}$.

The evolution of the baseline model is:

$$\begin{bmatrix} y^* \\ \check{z}^* \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} y \\ \check{z} \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} a + \begin{bmatrix} C_1 \\ K_2(\Delta)G \end{bmatrix} w^* + \begin{bmatrix} A_{12} \\ K_2(\Delta)D_2 \end{bmatrix} (z - \check{z}).$$

Under the benchmark model, the composite shock

$$\begin{bmatrix} C_1 \\ K_2(\Delta)G \end{bmatrix} w^* + \begin{bmatrix} A_{12} \\ K_2(\Delta)D_2 \end{bmatrix} (z - \check{z}) \tag{60}$$

is a normally distributed random vector with mean zero and covariance matrix

$$\Xi(\Delta) \doteq \begin{bmatrix} C_1 & A_{12} \\ K_2(\Delta)G & K_2(\Delta)D_2 \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & \Delta \end{bmatrix} \begin{bmatrix} C_1{}' & G'K_2(\Delta)' \\ A_{12}{}' & D_2{}'K_2(\Delta)' \end{bmatrix}$$

which can be factored as

$$\Xi(\Delta) = \tilde{C}(\Delta)\tilde{C}(\Delta)'$$

where $\tilde{C}(\Delta)$ has the same number of columns as the rank of $\Xi(\Delta)$. This factorization can be accomplished by first computing a spectral decomposition:

$$\Xi(\Delta) = U(\Delta)V(\Delta)U(\Delta)'$$

where $U(\Delta)$ is an orthonormal matrix and $V(\Delta)$ is a diagonal matrix with nonnegative entries on the diagonal. Partition $V(\Delta)$ by filling out its upper diagonal block with zeros:

$$V(\Delta) = \begin{bmatrix} 0 & 0 \\ 0 & V_2(\Delta) \end{bmatrix}.$$

The diagonal entries of $V_2(\Delta)$ are presumed to be strictly positive, implying that $V_2(\Delta)$ has the same dimension as the rank of $\Xi(\Delta)$. Partition $U(\Delta)$ conformably:

$$U(\Delta) = \begin{bmatrix} U_1(\Delta) & U_2(\Delta) \end{bmatrix}.$$

The matrix $\tilde{C}(\Delta)$ is then

$$\tilde{C}(\Delta) = U_2(\Delta)\left[V_2(\Delta)\right]^{1/2}.$$

Finally, let

$$\tilde{C}(\Delta) = \begin{bmatrix} \tilde{C}_1(\Delta) \\ \tilde{C}_2(\Delta) \end{bmatrix}$$

where $\tilde{C}_1(\Delta)$ has as many rows as there are entries in y and $\tilde{C}_2(\Delta)$ has as many entries as $\check{z}$.

We solve this game by simultaneously distorting the distribution of the composite shock defined in (60) instead of separately distorting the distributions of the components w^* and $(z - \check{z})$ of the composite shock. With this modification, we can solve the robust control problem as if there were no hidden Markov states. Let $\tilde{C}(\Delta)\tilde{u}$ denote the mean of the aggregate shock defined in (60). Write the single period objective as:

$$-\frac{1}{2}\begin{bmatrix} a' & y' \end{bmatrix}\begin{bmatrix} Q & P \\ P' & R \end{bmatrix}\begin{bmatrix} a \\ y \end{bmatrix} + \frac{\theta}{2}\tilde{u}'\tilde{u} = -\frac{1}{2}\begin{bmatrix} a' & \tilde{u}' & y' & \check{z}' \end{bmatrix}\Pi(\Delta)\begin{bmatrix} a \\ \tilde{u} \\ y \\ \check{z} \end{bmatrix}$$

where

$$\Pi(\Delta) = \begin{bmatrix} Q & 0 & P & 0 \\ 0 & -\theta I & 0 & 0 \\ P' & 0 & R & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Form an augmented control:

$$\tilde{a} = \begin{bmatrix} a \\ \tilde{u} \end{bmatrix}$$

and an augmented state:

$$\tilde{x} = \begin{bmatrix} y \\ \check{z} \end{bmatrix}.$$

Write the state evolution as:

$$\tilde{x}^* = \tilde{A}\tilde{x} + \tilde{B}(\Delta)\tilde{a}$$

where

$$\tilde{B}(\Delta) \doteq \begin{bmatrix} B_1 & \tilde{C}_1(\Delta) \\ B_2 & \tilde{C}_2(\Delta) \end{bmatrix}.$$

Write the discounted next period value function as

$$\beta V(\tilde{x}^*) = -\frac{\beta}{2}(\tilde{x}^*)'\Omega^*(\Delta^*)\tilde{x}^* - \beta\omega^*(\Delta^*).$$

Then the composite robust control is:

$$\begin{aligned} \tilde{a} &= -\left[\Pi_{11}(\Delta) + \beta\tilde{B}(\Delta)'\Omega^*(\Delta^*)\tilde{B}(\Delta)\right]^{-1}\left[\Pi_{12} + \beta\tilde{B}(\Delta)'\Omega^*(\Delta^*)\tilde{A}\right]\tilde{x} \\ &\doteq -\begin{bmatrix} \tilde{F}_1(\Delta) \\ \tilde{F}_2(\Delta) \end{bmatrix}\tilde{x} \end{aligned}$$

where $-\tilde{F}_1(\Delta)\tilde{x}$ is the control law for a and $-\tilde{F}_2(\Delta)\tilde{x}$ is the control law for u.

For the minimization part of the problem to be well posed, we require that θ be large enough that

$$\theta I - \beta\tilde{C}(\Delta)'\Omega(\Delta^*)\tilde{C}(\Delta)$$

is positive definite. The value function recursion is the Riccati equation:

$$\begin{aligned} &\Omega(\Delta) = \tilde{\Pi}_{22} + \beta\tilde{A}(\Delta)'\Omega^*(\Delta^*)\tilde{A}(\Delta) \\ &-\left[\Pi_{12} + \beta\tilde{B}(\Delta)'\Omega^*(\Delta^*)\tilde{A}\right]'\left[\Pi_{11}(\Delta) + \beta\tilde{B}(\Delta)'\Omega^*(\Delta^*)\tilde{B}(\Delta)\right]^{-1}\left[\Pi_{12} + \beta\tilde{B}(\Delta)'\Omega^*(\Delta^*)\tilde{A}\right]. \end{aligned}$$

The worst case covariance matrix for the composite shock is

$$\tilde{C}(\Delta)\left[I - \frac{\beta}{\theta}\tilde{C}(\Delta)'\Omega(\Delta^*)\tilde{C}(\Delta)\right]^{-1}\tilde{C}(\Delta)',$$

which is typically singular but larger than $\Xi(\Delta)$.

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