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Towards a Joint Characterization of Monetary Policy and the Dynamics of the Term Structure of Interest Rates

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Abstract:

The paper develops an empirical no-arbitrage Gaussian affine term structure model to explain the dynamics of the German term structure of interest rates from 1979 through 1998. In contrast to most affine term structure models two risk factors that drive the dynamics are linked to observable macroeconomics factors: output and inflation. The results obtained by a Kalman-filter-based maximum likelihood procedure indicate that the dynamics of the German term structure of interest rates can be sufficiently explained by expected variations in those macroeconomic factors plus an additional unobservable factor. Furthermore, we are able to extract a monetary policy reaction function within this no-arbitrage model of the term structure that closely resembles the empirical reaction functions that are based on the dynamics of the short rate only.

Keywords: affine term structure models, monetary policy rules, Kalman filter

JEL-Classification: E43, E58, G12

Non Technical Summary

In this paper, the dynamics of the German term structure of interest rates between 1979 and 1998 are investigated. This is done on the basis of an empirical model from the class of affine term structure models, which are based on the assumption of the absence of arbitrage opportunities in financial markets and which have become very popular in the finance literature. They model the dynamics of the term structure of interest rate on the basis of stochastic risk factors. Interest rates of different maturities are expressed as time-to-maturity-specific linear functions of that risk factors. However, in most cases these factors are left unobserved and, thus, have no explicit economic content. This paper, in contrast, provides an explicit link between the risk factors and two macroeconomic variables: the expected inflation gap and the expected output gap.

The explicit link results from the fact that within the affine term structure models the short-term interest rate is expressed as function of a constant plus the sum of risk factors each multiplied with an associated coefficient. This functional form equals the functional form of monetary policy rules which also gained popularity in macroeconomic research. They express the short-term interest rate (policy rate) as the sum of its equilibrium level and the reaction upon macroeconomic variables indicated by the reaction coefficient times the respective macroeconomic variable. From this formulation it follows that the latent risk factors of the affine term structure models could be matched with the macroeconomic variables that enter the monetary policy rules.

The paper connects the finance and the macroeconomic approaches and estimates an affine term structure model in which two factors are explicitly identified as the expected inflation gap and the expected output gap both of which have been proven to be significant in a monetary policy rule for the Bundesbank based on the short-rate only. The model is estimated with a Kalman filter technique based on interest rates whose maturities range from 1 month to 10 years. The results of two-factor as well as three-factor affine term structure models are presented. Especially the three-factor model with two factors being identified and the third factor being unobserved is able to describe the dynamics of the German term structure of interest rate quite well. The reaction coefficients with respect to the expected inflation gap and the expected output gap that are obtained are nearly identical to the ones that result from the traditional estimation of the monetary policy rule.

The results that are obtained by combining the two lines of research can be interpreted from two directions. From the view of the affine term structure literature it is shown that the risk factors have an explicit economic content. This results in a better understanding of the dynamics of the term structure. From the perspective of the macroeconomic literature on monetary policy rules we can draw the conclusion that the estimated monetary policy reaction coefficients that are extracted from the affine term structure model have the feature to be consistent with the absence of arbitrage in financial markets.

Nicht technische Zusammenfassung

In diesem Diskussionspapier erfolgt eine Untersuchung der Bestimmungsgründe der Dynamik der deutschen Zinsstruktur für den Zeitraum von 1979 bis 1998. Zur Erklärung der Zinsstrukturdynamik wird auf ein Modell aus der Klasse der so genannten Affinen Zinsstruktur-Modelle zurückgegriffen, welche sich in der finanzwirtschaftlichen Literatur großer Beliebtheit erfreuen. Diese Modelle basieren auf der expliziten Annahme der Arbitragefreiheit auf den Finanzmärkten und erklären die Dynamik der Zinsstruktur auf der Basis von so genannten stochastischen Risikofaktoren. Die Zinsen verschiedener Restlaufzeiten lassen sich dabei als restlaufzeitspezifische lineare Funktionen dieser Faktoren darstellen. In der Regel bleiben diese Faktoren jedoch unbeobachtet und haben keinen expliziten ökonomischen Gehalt. Im vorliegenden Diskussionspapier erfolgt allerdings eine explizite Identifikation bzw. Verbindung dieser Risikofaktoren mit makroökonomischen Variablen: der erwarteten Inflationslücke und der erwarteten Produktionslücke.

Die Möglichkeit zur Verbindung beider Ansätze ergibt sich aus dem Umstand, dass der kurzfristige Zins sich im Rahmen der Affinen Zinsstruktur-Modelle als Funktion einer Konstanten sowie der Summe aus den Risikofaktoren, welche jeweils multiplikativ mit einem Koeffizienten verknüpft sind, darstellen läßt. Diese funktionale Form entspricht der Form von geldpolitischen Reaktionsfunktionen, welche oftmals in der makroökonomischen Literatur zur Charakterisierung der Geldpolitik Verwendung finden. Hierbei wird die Dynamik des kurzfristigen Zinses, welcher von der Zentralbank gesetzt wird, erklärt durch dessen Gleichgewichtsniveau sowie der geldpolitischen Reaktion auf makroökonomische Variablen ausgedrückt durch die zu schätzenden Reaktionskoeffizienten in Bezug auf makroökonomische Variablen. Aus der Formulierung geldpolitischer Regeln folgt, dass es sich bei den latenten Risikofaktoren der Affinen Zinsstruktur-Modelle um jene makroökonomischen Variablen handeln sollte, die auch Eingang in die empirischen Reaktionsfunktionen finden.

Im Diskussionspapier erfolgt eine Schätzung der Dynamik der deutschen Zinsstruktur im Rahmen eines Affinen Zinsstruktur-Modells mit expliziter Identifikation zweier Risikofaktoren. Diese sind die erwartete Inflationslücke und die erwartete

Produktionslücke. Beide erweisen sich in einer vorgelagerten traditionellen Schätzung der Reaktionsfunktion der Deutschen Bundesbank als signifikant. Die Schätzung der Zinsstrukturdynamik erfolgt auf der Basis eines Kalman-Filter-Ansatzes unter Einbezug der Zinsen für die Restlaufzeiten eines Monats bis zu 10 Jahren. Dabei werden sowohl Zwei-Faktoren-Modelle als auch Drei-Faktoren-Modelle geschätzt. Vor allem das Drei-Faktoren-Modell, in welchem zwei der Faktoren explizit als erwartete Inflations- und Produktionslücke aufgenommen werden und der dritte Faktor unbeobachtet bleibt, erweist sich als geeignet, die Zinsstrukturdynamik zu erklären. Dabei ergeben sich nahezu die identischen Reaktionskoeffizienten wie in der traditionellen Schätzungen auf Basis des kurzfristigen Zinses.

Das Ergebnis, welches sich aus der Verbindung der beiden Ansätze ergibt, läßt sich aus zweierlei Sichtweise interpretieren. Aus Sicht der Affinen Zinsstruktur-Modelle erfolgt eine explizite ökonomische Fundierung der Risikofaktoren. Dies trägt zum besseren Verständnis der Zinsstrukturdynamik bei. Aus der makroökonomischen Sicht der Literatur zu geldpolitischen Regeln läßt sich schlußfolgern, dass die geschätzten geldpolitischen Reaktionskoeffizienten konsistent mit der Abwesenheit von Arbitragemöglichkeiten auf den Finanzmärkten sind.

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Towards A Joint Characterization of Monetary Policy and the Dynamics of the Term Structure of Interest Rates*)

1 Introduction

The characterization of monetary policy by empirical reaction functions that describe the short-term interest rate setting of central banks in response to the dynamics of a small set of macroeconomic variables, e.g. the inflation gap and the output gap, has become very popular in the macroeconomic literature in recent years. Such Taylor-type rules, however, only care about the dynamics of the short-term interest rate. They normally do not take into account the associated dynamics of longer-term interest rates, i.e. the term structure of interest rates. These dynamics are well described by a popular strand of the finance literature, that has also been growing rapidly in recent years. These are no-arbitrage models of the term structure of interest rates. The so-called affine term structure models (ATSM) are the most popular ones within this literature. They explain the dynamics of the term structure of interest rates by (mostly) unobserved stochastic (risk) factors, while yields of different maturities are connected by the absence of arbitrage opportunities.

Although a vast variety of affine term structure models exists due to the number of latent factors and the explicit formulation of their stochastic processes, they all share a common feature: in the single-factor case the only risk factor equals the short rate, whereas in multi-factor cases the short rate is a (additive) combination of multiple risk factors. Monetary policy rules share the same (functional) structure, once the risk factors are interpreted as macroeconomic variables. Therefore, the short-term interest rate is a critical point of intersection between the two lines of research. Together, the two perspectives suggest that understanding the manner in which central banks move the short rate (respectively, the policy rate) in response to fundamental macroeconomic

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variables should explain movements in the short end of the yield curve. With the consistency between long and short rates enforced by the no-arbitrage assumption, (expected) macroeconomic variations should amount for movements farther out of the yield curve as well. Combining the two lines of research could sharpen our understanding of the dynamics of the term structure of interest rates, respectively, the yield curve.

Understanding what moves the bond yields is important for at least four reasons. A first one is forecasting. When adjusted for risk, yields of long-maturity bonds represent expected values of average future short-term yields. Therefore, the yield curve contains information about the expected future path of the economy. A second reason is closely connected to monetary policy. Central banks are only able to move the short end of the yield curve via their interest rate decisions. However, what matters more for aggregate demand, and thus, for targeting inflation are the longer-term yields. A model of the dynamics of the yield curve helps to understand how movements at the short end translate into changes of longer-term yields. This knowledge helps central banks to conduct monetary policy. Debt policy is a third reason. When issuing new debt, governments need to decide about the maturity of the new bonds. Thus, they need an idea how the term structure of interest rates develops over time. A fourth reason is derivative pricing and hedging. Prices of complex securities, such as swaps, caps and floors, options on interest rates, and futures can be computed from a given model of the yield curve. Furthermore, banks need to manage the risk of paying short-term interest rates on deposits while receiving long-term interest rates on loans. Hedging strategies involve contracts that are contingent on future short rates, such as swap contracts. To compute appropriate strategies, banks need to know how the price of derivative securities depends on the state of the economy.

This paper combines the above mentioned two popular strands of the literature. We propose a model that is able to capture the dynamics of the German term structure of interest rates very well and additionally generates reaction coefficients in the short rate equation that are very close to the magnitudes that are observed in empirical models that abstract from arbitrage-free dynamics of the term structure of interest rates. However, in the formulation of the model we face a trade-off: we want to sufficiently characterize the dynamics of the term structure of interest rates but keep the model as tractable as possible, i.e., keeping the number of parameters as low as possible. For this reason we restrict the analysis to the class of constant volatility models with time-invariant term premia.

Several other papers have recently also combined the two lines of research. Ang and Piazessi (2003) estimate the dynamics of the US yield curve also based on macroeconomic factors as well as unobservable factors. They show that macroeconomic factors primarily explain movements at the short end and the middle of the yield curve while unobservable factors account mostly for movements at the long end. However, they do not jointly estimate the coefficients of the macroeconomic factors and the latent factors. They first estimate the coefficients of the monetary policy rule and estimate the remaining parameters in a second step holding the pre-estimated parameters constant. Rudebusch and Wu (2003) estimate a two-factor Gaussian term structure model for the U.S. Based on a subsequent OLS analysis they show that the extracted factors are correlated with factors that typically enter a monetary policy rule.

We depart from those ‘two step analyses’ by estimating an empirical model that jointly incorporates observable as well as unobservable factors. We also take into account the term structure dynamics up to a maturity of 10 years, while those studies take 6 years as the longest maturity. Our analysis draws on the study of Cassola and Luis (2003) who show that the dynamics of the German term structure from 1972 through 1998 can be well explained by a two-factor Gaussian model with time-invariant risk premia. We add explicit economic content to their model. Hördahl et al. (2003) also construct a joint model of German macroeconomic factors and the yield curve. However, they estimate the macroeconomic and the yield curve dynamics within the framework of the new neo-classical synthesis. Our approach lacks such a structural model but has the advantage to link macroeconomic and latent term structure factors more explicitly.

The paper is structured as follows. The subsequent section briefly characterizes the class of affine term structure models as well as the concept of the absence of arbitrage and its consequences for the pricing of bonds. Section 3 introduces the general multi-factor affine term structure model in discrete time (the Duffie Kan model) which nests most of the known affine term structure models. In section 4, this general model is tailored to a particular Gaussian version that serves as our empirical model. Section 5 relates the literature of monetary policy rules to the model sketched out before and relates the parameters to each other. The estimation procedure based on a Kalman filter and the results are presented in section 6. Finally, section 7 concludes.

2 An Arbitrage-Free Perspective of the Term Structure of Interest Rates

2.1 A Non-technical Characterization of Affine Term Structure Models

The term structure of interest rates can be characterized by affine term structure models.[1] These models are based on an explicit no-arbitrage condition in financial markets. The assumption of the absence of arbitrage opportunities seems quite natural for bond yields. Most bond markets are extremely liquid, and arbitrage opportunities are traded away immediately. Tractability is the main advantage of affine models: they assume bond yields to be affine (i.e., constant-plus-linear) functions of some state vector (the risk factors), so that models can easily be solved either in closed form or numerically using standard procedures.[2] The models of Vasicek (1977) and Cox, Ingersoll and Ross (or CIR, 1985) are the pioneers of the class of affine term structure models. Both focus on closed form solutions and show how the term structure of interest rates at a moment in time reflects regularities of interest rate movements over time. In the simplest versions of such models, the so-called one-factor models, the short-term interest rate is the single factor that drives the movements of the term structure.

The underlying diffusion process of the short rate is modeled as a Markov process, which means that its history contains no information about its future value that cannot be extracted from its current value. The Markov property is consistent with the so-called weak form of market efficiency, which generally says that extraordinary returns cannot be achieved by the use of the precise historical evolution in the price of a particular asset. The Markov process makes it possible that prices of zero coupon bonds of any maturity can also be written as a function of the short rate and their time to

1 See Maes (2004) and Piazessi (2003) for excellent overviews.

maturity. In this sense, the zero coupon bonds are derivative securities, i.e. securities deriving their value from the underlying short rate dynamics. Bond pricing can, thus, be performed along the well-known logic of derivative pricing including the Black and Scholes (1973) model of stock option pricing.

If the short rate follows a particular diffusion process and bond prices are a function of the short rate, then, by *Ito's Lemma*, the latter will also follow a diffusion process whose drift and variance can be characterized by the drift and variance of the short rate diffusion process.[3] Given a set of risky bonds, it is possible to design a risk-free self-financing portfolio that has to yield the instantaneously risk-free (short) rate within the interval *dt* in order to be consistent with the absence of arbitrage. This riskless portfolio is fully characterized in terms of the portfolio weights that have to be attached to the single assets. It turns out that the excess return per unit of risk for each asset has to be identical. This defines the market price of risk. In a setting with more than one risk factor, one market price of risk is associated with each factor. Formally, the existence of a unique (vector of the) market price(s) of risk requires the absence of arbitrage.

The market price of risk relates the expected return of a particular bond to its volatility, both of which can be expressed in terms of the drift and the volatility of the underlying short rate process (by *Ito's Lemma*). Expressing the market price of risk in terms of the latter derives the fundamental partial differential equation that each interest rate derivative has to satisfy in the absence of arbitrage: the so-called *term structure partial differential equation*. However, a unique solution for this term structure partial differential equation does not always exist. But it can be shown that an affine structure of bond prices guarantees a solution. This feature makes the class of affine term structure models particularly attractive.[4]

The logic that single-factor affine term structure models are based on the dynamics of the short rate indicates that the formulation of the nature of the diffusion

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[2] A function $f(\cdot)$ is defined to be affine if it is constant-plus-linear in its arguments (strictly speaking, linear would suffice). A univariate example would be: $f(x) = a + bx$, for all real parameters a and b.

[3] Ito's Lemma requires additionally that the bond pricing functions have to be twice differentiable. See Bolder (2001) for an in-depth illustration of the particular steps involved in the solution of ATSM.

[4] In particular, the affine structure decomposes the term structure equation into two separate differential equations that together form a so-called *Ricatti problem* that is quite easy to handle.

process for the short rate is crucial for any term structure model. The Vasicek (1977) model assumes that the short rate's volatility is constant, whereas in the CIR (1985) model the short rate's volatility is to be proportional to the square root of the short rate itself. This has the advantage of ruling out negative values of the interest rate because short rate volatility declines to zero when the short rate is zero. Thus, its future value is only determined by its drift which is positive. Both models have in common that the short rate exhibits mean reversion. A stochastic process that shows such dynamics is known as an *Ornstein-Uhlenbeck* process. Chan et al. (1992) provide empirical evidence that interest rates do indeed have the feature of mean reversion.

However, one-factor models have some unrealistic properties. First, they are not able to generate all the shapes of the yield curve that are observed in practice. For example, the one-factor Vasicek and CIR models can only produce increasing yield curves, decreasing yield curves and yield curves with a small hump (i.e, $\cap$-shaped). Typically, yield curves have one of these shapes. However, they can occasionally differ from these, e.g. the yield curve is sometimes decreasing for short maturities and then increasing for longer maturities. Such an inverse hump (i.e., $\cup$-shaped) or any other yield curve cannot be generated by one-factor models. Second, one-factor models do not allow for the twist of the term structure of interest rates, i.e. yield curve changes where short-maturity yields move in the opposite direction of long-maturity yields. This is because all yields are driven by a single factor, meaning that they have to be highly correlated. Intuitively, multi-factor models are much more flexible and are able to generate additional yield curve shapes and yield curve dynamics.[5]

In multi-factor models several, say k, observed or unobserved (risk) factors govern the movement of the term structure. The univariate diffusion process of the short rate is substituted by a multivariate diffusion process. Duffie and Kan (hereafter DK, 1996) establish the conditions that have to be fulfilled to still produce affine yield expressions to preserve the tractability of the models in the multidimensional case. The DK-framework has the advantage that it nests most of the existing term structure models. Among them are multi-factor versions of Vasicek (1977) and CIR (1985), as

[5] Empirical evidence strongly points towards multi-factor extensions of the model. See Dewachter and Maes (2000) and Dai and Singleton (2000) for a comparison of one-factor versus two- and three-factor model fits.

well as Longstaff and Schwartz (1992), Langetieg (1980) and Balduzzi et al. (1996 and 1998).[6] As in the one-factor case these models specify the stochastic process of the factors and derive the bond prices (or yields) as affine functions of the factors and the time to maturity. Since zero coupon bond prices and zero coupon yields are unambiguously tied together, yields (and thus the short rate) are also affine functions of the factors. Their dynamics are driven by the dynamics of the underlying risk factors.

The main (statistical) idea of affine term structure models, i.e., to explain the dynamics of the term structure of interest rates on the basis of few factors, is equal to the *modus operandi* of (purely statistical) factor models, such as the principal component analysis of yield changes that describe the covariance matrix of yield changes in terms of few factors that describe their common movement. For example, on the basis of a factor analysis Bühler and Zimmermann (1996) show that the dynamics of the German interest rate could well be described, i.e. about 90 percent of the variation, by three factors. Litterman and Scheinkman (1991) established the same result for the U.S. term structure within a principal component analysis.[7] The latter authors proposed the interpretation of this components in terms of 'level', 'slope' and 'curvature'. The major drawback of these analyses is that they are purely statistical and do not have any economic content. Affine term structure models start from the strong assumption of the absence of arbitrage and, thus, have an explicit economic content that puts restrictions on the cross-section and time series behavior of bond prices, respectively interest rates.

2.2 The Implications of the Absence of Arbitrage

The absence of arbitrage has crucial implications for the dynamics of any asset and, thus, for the term structure of interest rates. Harrison and Kreps (1979) and Harrison and Pliska (1981) prove that the assumption of the absence of arbitrage opportunities within a particular market is equivalent to the existence of a so-called pricing kernel (or stochastic discount factor). They also prove that the absence of arbitrage guarantees the existence of a risk-neutral probability measure, such that the price of any asset in time t that pays no dividend in time t+1 equals its time-t-expected

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[6] See Dai and Singleton (2000) for a categorization of affine term structure models. These authors also extensively discuss the restrictions that have to be fulfilled in the parameterization of the models in order be theoretically admissible and econometrically identified.

[7] Canabarro (1995) reports similar results for the US.

discounted value in t+1, where the discount factor is the risk-free rate and the expectations are formed under the risk-neutral measure. The risk-free probability can be converted to the data-generating measure (or physical measure) by the *Radon-Nikodym* derivative. The dynamics of the *Radon-Nikodym* derivative are driven by the dynamics of the market price of risk characterized above. The two concepts of the pricing kernel and the risk-neutral probability measure are mathematically equivalent and are jointly referred to as the "Fundamental Theorem of Asset Pricing".[8]

Our discrete-time model relies on the pricing kernel formulation. This essentially means that in the absence of arbitrage the price of any financial asset corresponds to the present value of its expected future cash flow with the present value being obtained by applying the positive stochastic discount factor (the pricing kernel). In the case of zero coupon bonds whose future cash flows only correspond to their price in the next period, we can establish the following pricing equation:

$$P_{n,t} = E_t\left[M_{t+1}P_{n-1,t+1}\right], \tag{2.1}$$

where the expectations are taken under the physical probability measure. P represents the price of the zero coupon bond with n denoting its time to maturity. M is the stochastic discount factor which is also known as the price generator, since prices grow from it. In any arbitrage-free environment, there exists a unique positive random variable that satisfies expression (2.1).[9] Arbitrage opportunities are ruled out by applying the same discount factor to all bonds. An arbitrage opportunity is any zero-net-investment strategy that guarantees a positive payoff in some future state of the world with no possibility of a negative payoff in all other future states of the world.[10] Solving

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8 See Maes (2004, p. 11 ff) on the exact nature of this equivalence in continuous-time models and Backus et al. (1998a) for the discrete-time models.

9 See Harrison and Kreps (1979) and Harrison and Pliska (1981) for this fundamental result.

10 Intuitively, the equivalence of the existence of a unique positive pricing kernel and the absence of arbitrage opportunities can be explained as follows. An arbitrage project would be a trading strategy that satisfies the following condition: $V_t^y < 0$ and $V_{t+1}^y \geq 0$, with *y being a vector* representing the units held of each asset in a portfolio between time t and $t+1$. This relation basically means that the portfolio has a negative initial value (so that the investor receives money when initiating the strategy in time t), while its value (payoff) in time $t+1$ is non-negative no matter how the world evolves between the two dates. Any rational investor would want to invest in this strategy. Applying (2.1) to this portfolio yields: $V_t^y = E_t\left[M_{t+1}V_{t+1}^y\right]$. If the pricing kernel is positive and V_{t+1}^y is strictly positive, it

forward the basic pricing equation (2.1) by the *law of iterated expectations* and noting that the bond pays exactly one unit at maturity ($P_{0,t+n}$=1) yields:

$$P_{n,t} = E_t\left[M_{t+1} \dots M_{t+n}\right] = E_t\left[\prod_{i=1}^{n} M_{t+i}\right], \qquad (2.2)$$

so that a model of bond prices could also be expressed as a model of the evolution of the pricing kernel. It follows that we can model $P_{n,t}$ (and, thus, associated bond yields) by modeling the stochastic process of $M_{t+\mathrm{i}}$. The bond prices (and, thus, bond yields) are a function of those state variables that are relevant for forecasting the process of the pricing kernel.

In consumption-based equilibrium models, the pricing kernel represents the marginal rate of substitution between present and next period's consumption, i.e. the discounted ratio of marginal utilities of consumption, of a particular representative agent valued at her optimal consumption rate.[11] Since the purpose of financial assets is to allow agents to shift consumption across time and different states, it should come at no surprise that a measure for the market-wide pricing information can be captured by the marginal rate of substitution in consumption. Furthermore, the pricing kernel is unique in the case financial markets are complete. A market is complete if all relevant risks can be hedged by forming portfolios of the traded financial assets.[12]

Arbitrage free models can also be considered as equilibrium models, i.e. only equilibrium prices of financial assets are determined. A market is said to be in an equilibrium if it clears in the sense that demand equals supply, and every investor has picked a trading strategy in the financial asset under consideration that serves his

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must be that V_t^y has to be strictly positive, too. Consequently, if a positive pricing kernel exist, arbitrage as defined above is ruled out.

11 This can be shown by considering the intertemporal choice of an investor who maximizes the expectation of a time-separable utility function: $U_t = E_t\left[\sum_{j=0}^{\infty} \Gamma^j u(c_{t+j})\right]$, which leads to the optimal consumption plan: $u'(c_t) = \Gamma \cdot E_t\left[(1+i_t)\cdot u'(c_{t+1})\right] = \Gamma \cdot E_t\left[\left(P_{n-1,t+1} \big/ P_{n,t}\right)\cdot u'(c_{t+1})\right]$, where i_t is the return of an asset that the investor can freely trade and the second equality comes from our bond considered above, since we have as its gross return: ($P_{n-1,t+1}/P_{n,t}$) = (1+i_t). Comparing this to (2.1) it follows that the pricing kernel is $M_{t+1} = \Gamma \cdot u'(c_{t+1})/u'(c_t)$. Note that this also means that the pricing kernel is always positive since marginal utilities are always positive.

12 This essentially means, for example, that options can be artificially created by a suitable buy and sell strategy in the underlying asset.

preferences and budget constraint. An arbitrage is a trading strategy that generates a riskless profit. If an investor has the opportunity to invest in an arbitrage project, he will surely do so, and hence will change his original trading strategy. In other words, a market that allows for arbitrage is not in an equilibrium. For our purpose this means, when searching for equilibrium prices we can exploit no-arbitrage conditions.

We can also employ the fundamental pricing equation (2.1) to characterize the compensation for risk that an investor demands for holding a risky bond. If we denote the nominal gross return of an asset ($P_{n-1,t+1}/P_{n,t}$) as ($1+i_t$) we can rewrite (2.1) and get:[13]

$$1 = E_t\left[\frac{P_{n-1,t+1}}{P_{n,t}} M_{t+1}\right] = E_t\left[(1+i_t)M_{t+1}\right] = E_t\left[(1+i_t)\right]E_t\left[M_{t+1}\right] + Cov_t\left[i_t, M_{t+1}\right]. \tag{2.3}$$

It follows that:

$$E_t\left[(1+i_t)\right] = \frac{1}{E_t\left[M_{t+1}\right]}\left(1 - Cov_t\left[i_t, M_{t+1}\right]\right). \tag{2.4}$$

Since the covariance term has to be zero for a risk-free asset, its rate of return has to satisfy

$$1 + i_t^f = \frac{1}{E_t\left[M_{t+1}\right]}. \tag{2.5}$$

The risk free rate is often referred to as the short rate. However, this means that it is only ‘instantaneously riskless’ because future levels are not known. The short rate is strictly speaking a zero maturity interest rate, i.e. the interest rate of a bond that matures in the next instant. Since the shortest maturity we will consider in our empirical exercise is one month, we will think of the one-month rate as the risk-free short rate. Combining (2.4) and (2.5) we get an expression for the excess return of any risky asset over the risk-free rate, in other words, its risk premium:

$$E_t\left[i_t\right] - i_t^f = -(1 + i_t^f)Cov_t\left[i_t, M_{t+1}\right]. \tag{2.6}$$

[13] See Campbell et al. (1997, p. 294).

The previous equation basically demonstrates that risk premia originate from the co-variation of its return with the pricing kernel. When the return dynamics of an asset covary negatively with the pricing kernel dynamics, the risk premium is positive, and *vice versa*. The minus sign in (2.6) means that investors are willing to accept an expected rate of return below the risk free rate on such securities that tend to have high payoffs when the pricing kernel is higher. This is best understood if we rely on the interpretation of the pricing kernel as the ratio of marginal utilities of consumption. An investor will demand a higher compensation for an asset that behaves cyclically than for an asset that behaves countercyclically, since the former bears higher risk in terms of consumption variability, whereas the latter can be considered as an insurance against consumption risk.[14]

3 A General Affine Term Structure Model in Discrete Time

As already mentioned, Duffie and Kan (1996) present a framework that formally preserves the affine property of term structure models. Most existing models can thus be interpreted as a particular parameterization of this framework. Originally the DK-framework was formulated in continuous time. This way of formulation has proven to be very elegant by using well-known results of differential stochastic calculus. However, affine term structure models can also be formulated in discrete time.[15] Since we focus on econometric testing of the model and its empirical implications, we also rely on a model in discrete time. This might be less elegant but avoids the pitfalls of estimating a continuous-time model with discrete-time data.[16]

The DK model has been reformulated by Backus et al. (1998a/b) into discrete time and is based on the fundamental result of the existence of a unique pricing kernel characterized above. Equation (2.1) demonstrates that in an arbitrage-free environment a model that is intended to explain the development of asset prices consists of a description of the development of the pricing kernel which, in turn, is driven by the

[14] When consumption growth is high the marginal utility of consumption (and thus the pricing kernel) is low. If returns are negatively correlated with the pricing kernel, high returns are associated with states of high consumption. A risk premium has to be paid for such assets because they provide more wealth when it is less needed, respectively they provide less wealth when it is most needed.

[15] See Backus et al. (1998a) for an extensive exposition. Sun (1992) explores the relation between discrete-time and continuous-time models more generally.

process of the underlying risk factors. In particular, the k-dimensional vector of risk factors, Z, satisfies the following stochastic process:[17]

$$Z_{t+1} = \left(I - \Phi\right) \cdot q + \Phi \cdot Z_t + V(Z_t)^{1/2} e_{t+1} \qquad (3.1)$$

where I is an identity matrix and the matrix Φ has positive diagonal elements between zero and one in order to ensure that the factors are stationary. q is the long-run mean of the risk factors. Thus, the risk factors are governed by a discrete-time *Ornstein-Uhlenbeck* process. The independent shock term is normally distributed with $e_t \sim N(0, I)$. Finally, $V(Z_t)$ is the variance-covariance matrix of the random shocks and is defined as a diagonal matrix with the elements $v_j(Z_t) = a_j + b'_j Z_t$, where b_j has nonnegative elements.[18] Thus, the factors have an affine volatility structure, which is a generalization of the square-root structure of the CIR model. Moreover, the factors are allowed to be correlated. The square-root process of the risk factor requires that the volatility function $v_j(Z)$ has to be positive, which places particular restrictions on the parameters.[19]

To derive an affine yield model, the distribution of the stochastic discount factor is assumed to be conditionally log-normal. In addition to providing model tractability, this assumption keeps the discount factor positive and unique. Following Backus et al. (1998a), the negative of the log of the pricing kernel ($m_{t+1} \equiv \log\left[M_{t+1}\right]$) takes the form:

$$-m_{t+1} = d + g' \cdot Z_t + x_{t+1},$$

where d is a constant and g is a parameter vector. The innovations to the risk factors and the pricing kernel may be correlated. To capture this we write:

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16 See A?t-Sahalia (1996). He points out that the approximation of a continuous time process by discretization methods is hard to justify even for daily data.

17 The following model is covered in greater detail in Cassola and Luis (2003), who also take the DK-framework as the benchmark for their model.

18 In the subsequent notation the subscripts j generally indicate a particular element within the respective matrix which has no subscript attached. Throughout the paper the subscript j = 1 ... k indicates the number of risk factors, and thus, determines the size of the matrices and vectors.

19 In a continuous-time framework the dynamics of the risk factors are generally expressed as an Ito diffusion process. Expression (3.4) is its counterpart in discrete time. Formally, (3.4) can be derived via a *Euler discretization* (see Bolder, 2001, p. 51).

$$x_{t+1} = l' \cdot V(Z_t)^{1/2} e_{t+1} + h_{t+1}.$$

The presence of an uncorrelated shock h_{t+1} will only affect the average level of the term structure and not its average slope nor its time-series behavior. In order to simplify notation it is common to drop it (Campbell et al. 1997), so that we have:

$$-m_{t+1} = d + g' \cdot Z_t + l' \cdot V(Z_t)^{1/2} e_{t+1}, \tag{3.2}$$

where l is the vector that governs the correlation between innovations in the risk factors (state variables) and the pricing kernel: risk, in other words.

According to the affine formulation of bond prices it is assumed that bond prices are exponential affine functions of the risk factors:[20]

$$P_{n,t} = \exp\left(-A_n - B_n' \cdot Z_t\right), \tag{3.3}$$

where, again, the second subscript n denotes the time to maturity at time t. The parameter A_n and the vector of parameters B_n are to be estimated in order to determine bond prices. Both are 'time to maturity-related' constants. The latter is commonly referred to as the vector of factor loadings because it measures the impact of a shock to the risk factors on the bond prices. In case of zero coupon bonds the values A_0 and B_0 have to be equal to zero, because at the time of their maturity (n = 0) they pay by definition exactly the face value of one unit. Thus, the log of the price of a maturing bond (denoted by a lower case p) has to be zero. The general form of the relation is:

$$-p_{n,t} = A_n + B_n' \cdot Z_t. \tag{3.4}$$

Since nominal yields and prices of zero coupon bonds are unambiguously linked, nominal yields of zero coupon bonds can be easily computed as

$$i_{n,t} = -\frac{p_{n,t}}{n} = \frac{A_n}{n} + \frac{B_n'}{n} Z_t. \tag{3.5}$$

[20] This particular formulation ensures the property that log bond prices, and hence bond yields, are linear (affine) in the state variables. This ensures the desired joint log-normality of bond prices with the stochastic discount factor.

Equation (3.5) states that we can model the dynamics of the whole term structure if we are able to estimate the parameters A_n and the vector B_n for all values of *n*. In order to solve for their dynamics, we need to employ a well-known statistical result for the distribution of a log value of any variable *X*. This result states that if $\log X \sim N(m, s^2)$ then $\log E(X) = m + \frac{s^2}{2}$.

Applying this to the fundamental pricing relation in (2.1) expressed in logs together with eqs. (3.1), (3.2), (3.4), the known (zero) values A_0 and B_0, and the assumption of the independence of shocks the values of A_n and B_n will follow the subsequent recursive restrictions:[21]

$$A_{n+1} = A_n + d + B_n'(I - \Phi)q - \frac{1}{2}\sum_{j=1}^{k}\left(l_j + B_{j,n}\right)^2 \cdot a_j \tag{3.6}$$

$$B_{n+1}' = \left(g' + B_n'\Phi\right) - \frac{1}{2}\sum_{j=1}^{k}\left(l_j + B_{j,n}\right)^2 \cdot b_j', \tag{3.7}$$

where k is the number of risk factors that are modeled.

From (3.5) through (3.7) we can obtain the following expression for the one-period interest rate (short rate):

$$\begin{aligned} i_{1,t} &= d - \frac{1}{2}\sum_{j=1}^{k} l_j^2 \cdot a_j + \left(g' - \frac{1}{2}\sum_{j=1}^{k} l_j^2 \cdot b_j'\right) \cdot Z_t \\ &= d + g' \cdot Z_t - \frac{1}{2} l' V(Z_t)\, l, \end{aligned} \tag{3.8}$$

where the second equality stems from the fact that the variance-covariance matrix is defined as a diagonal matrix.[22]

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[21] See Appendix A.1 for the detailed derivation as well as Campbell, Lo and MacKinlay (1997, ch. 11) for the solution technique.

[22] This expression for the short rate is entirely consistent with the fundamental relationship between the risk-free rate and the pricing kernel in (2.5), once we take the one-period interest rate as the risk-free rate. Expressing (2.5) in logs and employing the usual approximation for the logarithmic form of the gross rate, $\log(1+i) \cong i$, we have: $\log E_t\left[M_{t+1}\right] = -i_{1,t}$. Given the above stated statistical result of the

We can also calculate the term premia as the difference between the 'holding period return' and the one-period interest rate (see Appendix A.2):

$$\Lambda_{n,t} = E_t p_{n,t+1} - p_{n+1,t} - i_{1,t}$$

$$= -\sum_{j=1}^{k} \left(l_j B_{j,n} + \frac{B_{j,n}^2}{2} \right) \cdot a_j - \sum_{j=1}^{k} \left(l_j B_{j,n} + \frac{B_{j,n}^2}{2} \right) \cdot b_j' \cdot Z_t \,.$$

Given the structure of the variance-covariance matrix this is equivalent to:

$$\Lambda_{n,t} = -l' \cdot V(Z_t) \cdot B_n - \frac{B_n' \cdot V(Z_t) \cdot B_n}{2} \,. \tag{3.9}$$

This relation basically says that the expected excess log return is the sum of a risk premium term and a *Jensen's Inequality* term in the own variance because we are working in logs. The term premium is governed by the vector l. A negative (positive) sign leads to a positive (negative) bond risk premium. This can be reasoned as follows. Consider a positive shock e_{t+1} which increases the state variable. According to (3.4) this lowers all bond prices and drives down bond returns. When l is positive, the shock also drives down the log value of the pricing kernel, which means that bond returns are positively correlated with the pricing kernel. As explained above, this correlation has a hedge value, so that risk premia on bonds are negative. The same reason applies to the case when l has negative sign, which leads to positive risk premia. l is usually known as the vector of market prices of risk. In order to facilitate a normally (i.e., positively) sloped yield curve, at least one parameter in the vector l has to be sufficiently negative.

4 A Gaussian Model for the German Term Structure of Interest Rates

This section proposes a particular parameterization of the general model sketched out above. The parameter choice is driven by the following trade-off. On the one hand,

log-value of an expectation of a log-normal distributed variable, the assumption of the determination of the pricing kernel in (3.2) together with (3.8) yields the above stated expression for the log of the

the model has to be sufficiently flexible to be able to characterize the dynamics of the German term structure of interest rates. This might call for a fully fledged model with a high number of parameters to be estimated. On the other hand, our interest lies in keeping the number of parameters as low as possible, in order to keep the model empirically tractable and also to assign specific economic interpretation to the estimated parameters. In order to resolve this trade-off, we draw on the study of Cassola and Luis (2003). These authors recently demonstrated that the dynamics of the German term structure of interest rates can be well explained within a constant volatility (or Gaussian) two-factor affine term structure model. Close to their parameterization our choice of parameters is:

$$
\begin{aligned}
& q_j = 0 \\
& \Phi = diag(j_1 \ldots j_k) \\
& a_j = s_j^2 \qquad\qquad (4.1) \\
& b_j = 0 \\
& d = \bar{d} + \frac{1}{2}\sum_{j=1}^{k} l_j^2 s_j^2 .
\end{aligned}
$$

The long-run mean of the risk factors is set equal to zero because of our particular identification assumption of the risk factors that follows below.[23] The matrix of the mean reversion parameters is assumed to be diagonal so that the dynamics of a particular risk factor only depend on its own current value relative to its long-run mean. The third and fourth definition in (4.1) constitute a Gaussian model with constant volatility of the risk factors.[24] The last definition in (4.1) is intended to yield a particular

expectation of the pricing kernel, meaning that the general formulation of the model is consistent with the basic pricing relation in an arbitrage free environment discussed in section 2.

23 There is another technical justification for this as well. Backus et al. (1998a) show for the single-factor case of the constant volatility model the parameters d and q cannot be identified separately, so that one of them could effectively be dropped. Setting either d or q to zero implies identical asset prices. However, in the general setup of the model this is not the case.

24 Note that the constant volatility implies that the term premia in (3.9) only depend on the time to maturity but not on time anymore. Due to the constant volatility formulation Backus et al. (1998a, p.

formulation for the short-rate equation.[25] In contrast to Cassola and Luis (2003) we limit the number of restrictions put on the parameters and admit the parameter vector g to be determined by the data.

The parameterization yields the following equation for the dynamics of the pricing kernel:

$$\begin{aligned} -m_{t+1} &= \bar{d} + \frac{1}{2} l' s\, s' l + g' \cdot Z_t + l' s\, e_{t+1} \\ &= \bar{d} + \sum_{j=1}^{k} \left(\frac{l_j^2}{2} s_j^2 + g_j z_{j,t} + l_j s_j e_{j,t+1} \right). \end{aligned} \tag{4.2}$$

The vector of risk factors follows a first-order autoregressive process:

$$Z_{t+1} = \Phi \cdot Z_t + s\, e_{t+1},$$

or for individual risk factors j:

$$z_{j,t+1} = j_j z_{j,t} + s_j e_{j,t+1}. \tag{4.3}$$

Our particular parameterization yields the following recursive restrictions for the factor loadings (see again Appendix A.1 for the derivation):

$$A_{n+1} = A_n + \bar{d} + \frac{1}{2} \sum_{j=1}^{k} \left[l_j^2 s_j^2 - \left(l_j s_j + B_{j,n} s_j \right)^2 \right] \tag{4.4}$$

$$B_{j,n+1} = \left(g_j + B_{j,n} j_j \right). \tag{4.5}$$

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23) redefine the market prices of risk as $\tilde{l}_j \equiv l_j s_j$. This is in line with Duffie and Kan (1996) who also generally assume that the market prices of risk for factor j is proportional to its standard deviation. In equation (3.2) we would than substitute: $\tilde{l}' \equiv l' V(Z_t)^{1/2}$. Rather than following this approach we do not make this redefinition. This has the advantage to estimate both parameters and their standard errors independently. Cassola and Luis (2003) also do not redefine the prices of risk in writing down their model, but only provide estimates of $l_j s_j$, which they interpret as the market prices of risk. However, following this would not change our results, since we can also calculate the value $l_j s_j$ from our independent estimates.

From the recursive restrictions (4.4) and (4.5) together with (3.5), and the fact that $A_0 = B_0 = 0$, we can get an expression for the one-period interest rate:

$$i_{1,t} = \bar{d} + \sum_{j=1}^{k} g_j z_{j,t} \quad . \tag{4.6}$$

The one-period (or short-term) interest rate is determined by a constant and the sum of time-varying risk factors multiplied by related (constant) coefficients. Examining (4.6) and interpreting the one-period interest rate as the policy rate of a central bank, one immediately realizes its equivalence to the popular class of empirical monetary policy rules, in which the risk factors and their associated coefficients would have very specific economic meanings. In particular, the constant would be the equilibrium level of the nominal short-term interest rate, the g_j's would resemble the reaction coefficients of the central bank and the risk factors would display macroeconomic variables upon which the central bank reacts. From an empirical point of view equation (4.6) potentially allows us to jointly estimate a monetary policy reaction function together with the arbitrage-free dynamics of the term structure of interest rates. However, the crucial point for this exercise is the (empirical) identification of the risk factors.[26] This is the aim of the subsequent section 5.

5 The Monetary Policy View of the Short Rate Dynamics

Since the influential paper by Taylor (1993), it has become common practice to model monetary policy as a simple feedback or instrument rule. These rules link the short term policy rate to measures of the output gap and the inflation gap:

$$i_{1,t} = p_t + \bar{r} + g_p (p_t - p^*) + g_y (y_t - \bar{y}_t) \quad , \tag{5.1}$$

where i_1 is the policy rate, p is inflation with the (*) indicating its target value. The equilibrium real interest rate is $\bar{r}$ and y is an output measure with the upper bar

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[25] In particular, this variance term only arises because we are working in logs (*Jensen's Inequality*). The normalization lets us get rid of this term.

[26] Cassola and Luis (2003) parameterize the model such that the one-period rate is just the sum of risk factors. In the two-factor case they notice that this equals the *Fisher equation*, where the nominal interest rate is the sum of the real interest rate plus expected inflation.

indicating its equilibrium level. The coefficients g_p and g_y measure the strength of the reaction to the inflation gap, respectively, the output gap. A comparison of (5.1) and (4.6) reveals the same expression once the sum of the equilibrium real rate and the actual inflation rate reflect the $\bar{d}$ and the (two) risk factors are interpreted as the inflation and the output gap.

Clarida et al. (1998) generalize the formulation in (5.1) to a class of explicitly forward-looking instrument rules. They further allow for the effect that central banks smooth their changes in the policy rate, so that the actual rate adjusts only partially to its target rate.[27] The empirical representation of the reaction function becomes:

$$i_{1,t} = (1-r)\left(\bar{r} + E(p^{*}_{t+q}\,|\,I_t) + g_p E(p_{t+q} - p^{*}_{t+q}\,|\,I_t) + g_y E(y_{t+l} - \bar{y}_{t+l}\,|\,I_t)\right) + r \cdot i_{1,t-1} + n_t, \quad (5.2)$$

where r represents the smoothing parameter, E is the (conditional) expectation operator given the information set I_t, q and l are the horizons of the forward-looking behavior, and n is a composite error term that comprises exogenous shocks to the policy rate. Comparing (5.2) and (4.6) would then relate two risk factors with the expected inflation gap, the expected output gap as well as a third risk factor that comprises the effect of interest rate smoothing behavior and other (stochastic) shocks to the interest rate.

In order to explicitly combine the forward-looking monetary policy rule representation with the arbitrage-free model of the term structure, we need to characterize the dynamics of the inflation and the output process. In particular, we assume that the inflation gap and the output gap both follow an AR(1) process:

$$\tilde{p}_{t+1} = c_p \cdot \tilde{p}_t + u_{t+1}, \quad (5.3)$$

$$\tilde{y}_{t+1} = c_y \cdot \tilde{y}_t + u_{t+1}, \quad (5.4)$$

[27] However, the statistical significance of a lagged policy rate can have alternative interpretations, e.g. a serial correlation of shocks. See Rudebusch (2002) for a critical discussion.

where the $\tilde{x}$ -notation reflects the gap of the respective variable.[28] Forward iteration of (5.3) yields:

$$\tilde{p}_{t+q} = c_p^{\,q} \cdot \tilde{p}_t + \sum_{i=1}^{q} c^{\,q-i} u_{t+i} \quad . \tag{5.5}$$

If we match one of the risk factors in time t with the expected inflation gap $t+q$ periods ahead

$$z_{p,t} = E\left[\tilde{p}_{t+q}\right] = c_p^{\,q} \cdot \tilde{p}_t \quad , \tag{5.6}$$

the law of motion of this (inflation) risk factor becomes

$$z_{p,t+1} = E\left[\tilde{p}_{t+q+1}\right] = c_p \cdot \tilde{p}_{t+q} = c_p \left(c_p^{\,q} \cdot \tilde{p}_t\right) + c_p \left(\sum_{i=1}^{q} c^{\,q-i} u_{t+i} \right). \tag{5.7}$$

Combining (5.6) and (5.7) with the law of motion for the risk factors from the affine term structure model in (4.3) it follows for the identification of parameters that:

$$\begin{aligned} j_p &= c_p \\ s_p e_{t+1} &= c_p \cdot \sum_{i=1}^{q} c_p^{\,q-i} u_{t+i} . \end{aligned} \tag{5.8}$$

From (5.6) it follows that the actual inflation gap and the actual (inflation) risk factor are connected as follows:

$$\tilde{p}_t = \frac{1}{(j_p)^q} \cdot z_{p,t} \quad . \tag{5.9}$$

By the same token the relationship between the actual output gap and the (output) risk factor is:

[28] This processes might seem too simplistic, but we refer the reader to the empirical part, where this assumption turns out to produce quite reasonable results. In addition, since we link the risk factors whose dynamics are modeled through equation (4.3) to the macroeconomic factors, consistency requires that we assume the same autoregressive process.

$$\tilde{y}_t = \frac{1}{(j_y)^l} \cdot z_{y,t} \quad . \tag{5.10}$$

Equations (5.9) and (5.10) follow from the generalized monetary policy rule in (5.2) and display the general identification of two risk factors with observable macroeconomic variables given the assumption of an AR(1) behavior of the latter.[29] Equations (5.9) and (5.10) when combined with (3.5) provide us with the possibility to jointly estimate the yields and the two macroeconomic factors as functions of the risk factors.

6 Estimation Procedure and Results

6.1 The Data

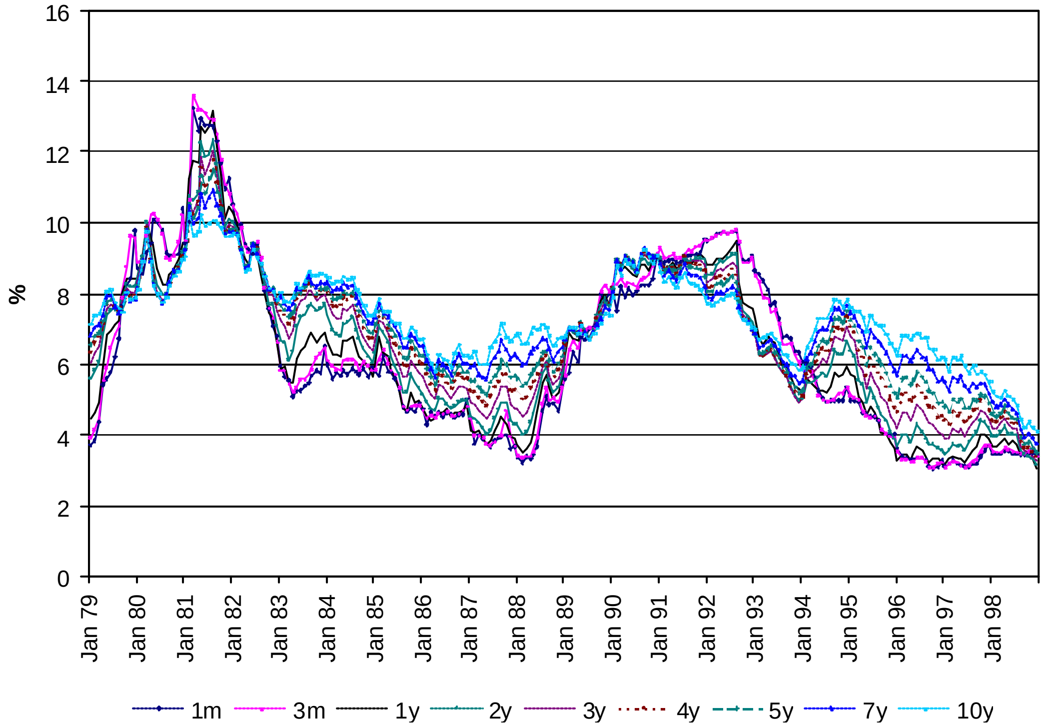
We focus on German data. The data are monthly and cover the period from 1:1979 to 12:1998, This period represents what we might call the 'Bundesbank regime'. The sample starts in 1979 because this represents the starting of the European Exchange Rate Mechanism and ends in 1998, when the European Central Bank took over responsibility for monetary policy in the euro area. The estimates of the German yield curve are based on the parameters of the Svensson (1994) smoothing technique as made available by the Bundesbank. The data set comprises monthly averages of nine daily spot rates for the following maturities (in month): 1, 3, 12, 24, 36, 48, 60, 84 and 120. Because of the somewhat erratic behavior we substitute the 'Svensson estimates' of the one month and three months interest rates by the observed German interbank rates.[30] Figure 1 presents the time series properties of the yields (in panel (a)) and the average yield curve over the sample period (panel (b)), while Table 1 presents the summary statistics.

[29] The Taylor rule in (5.1) is the special case with $q = l = 0$, so that: $\tilde{p}_t = z_{p,t}$ and $\tilde{y}_t = z_{y,t}$. In this particular case the risk factors and the two observable macroeconomic variables match on a one to one basis and we need no explicit assumption about the dynamics of the macroeconomic variables.

[30] We consider the money market rates as a good proxy, because the default risk on loans in the German money market can be considered to be quite low. In general, money market rates apply for unsecured loans between financial institutions and hence reflect a default risk which lead money market rates to be higher than similar bond market rates.

Figure 1: Term Structure Dynamics 1979-1998

(a)



b)

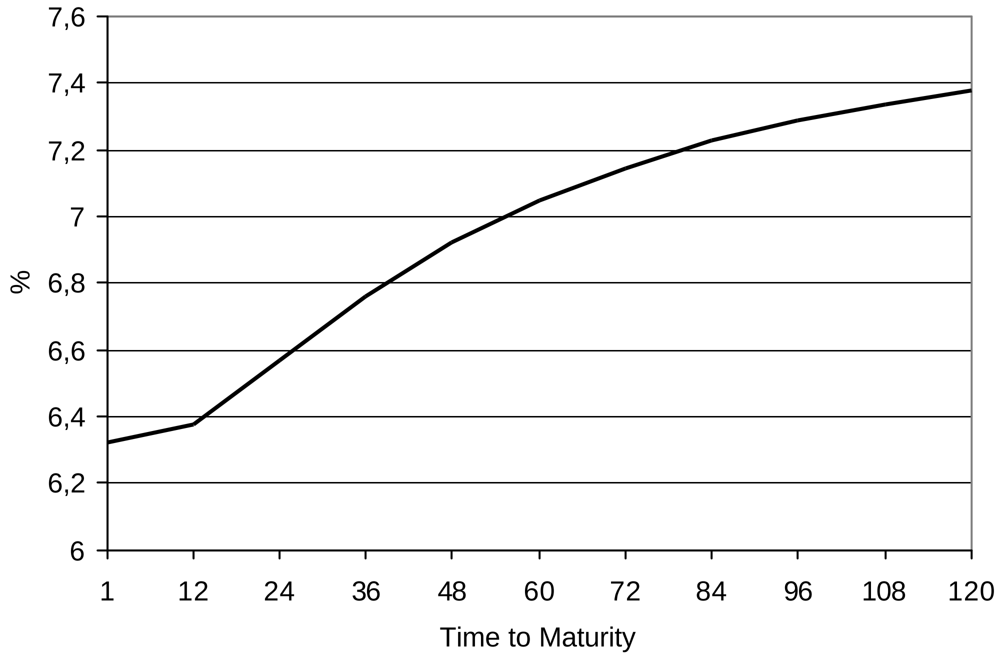


Table 1: Statistical Properties of German Government Bond Yields 1979 – 1998

Maturity	1	3	12	24	36	48	60	84	120
Mean	6.321	6.343	6.376	6.569	6.762	6.921	7.048	7.226	7.376
St.Dev	2.499	2.539	2.30	2.078	1.899	1.746	1.616	1.416	1.215
Skewness	0.604	0.599	0.544	0.379	0.316	0.287	0.259	0.167	0.031
Kurtosis	2.487	2.537	2.636	2.408	2.398	2.472	2.571	2.755	2.955
Autocorr.	0.982	0.984	0.982	0.980	0.978	0.976	0.973	0.968	0.960
Correlation Matrix									
1	1.0000	0.9967	0.9724	0.9399	0.9132	0.8914	0.8715	0.8319	0.7678
3		1.0000	0.9826	0.9543	0.9294	0.9084	0.8890	0.8500	0.7870
12			1.0000	0.9902	0.9755	0.9606	0.9453	0.9119	0.8552
24				1.0000	0.9959	0.9872	0.9760	0.9489	0.9009
36					1.0000	0.9974	0.9909	0.9708	0.9307
48						1.0000	0.9980	0.9850	0.9525
60							1.0000	0.9938	0.9689
84								1.0000	0.9899
120									1.0000

Table 1 reveals that yields are highly persistent with monthly autocorrelations above 0.96 for all maturities. The volatility drops with increasing maturity. Also yields are highly correlated along the yield curve but the correlations are not equal to one. This suggests that non-parallel shifts of the yield curve are an important feature. Indeed, the yield curve frequently changed its slope, shape and curvature and even periods of inverse yield curves can be identified. Therefore, a one-factor model is insufficient for explaining the dynamics of the German term structure of interest rates.

The data on the inflation gap is constructed using the difference of the consumer price index and the so-called price norm that has been announced by the Bundesbank on a yearly basis. This could be interpreted as a kind of an inflation target, although the Bundesbank did not explicitly target inflation but the money growth rate. Nevertheless, the inflation values were used as inputs into the derivation of the money growth target and price stability was the final target of the Bundesbank. Whenever the Bundesbank announced ranges instead of values of the price norm we opt for the middle of the range.[31]

[31] Until 1984, the price norm, or price assumption, reflected the Bundesbank's view of the "unavoidable" level of inflation, while from 1985 onwards, it was defined as the maximum rate of inflation to be tolerated over the medium term. Conceptually, the price norm should refer to the GNP/GDP deflator rather the CPI, since it was related to the price term in the quantity equation. However, we believe it is a good approximation for the implicit target of the consumer price inflation.

The output gap is based on the monthly index of industrial production. Although being aware that industrial output is more volatile than, say, GDP, we follow the usual practice and opt for this variable in order to obtain monthly figures. The output gap is constructed on the basis of a linear trend. In order to avoid starting point and end point problems we computed the trend between the longer period from 1970 through 2003.[32]

6.2 Estimation of Monetary Policy Rules

We first present the results of standard regressions of monetary policy reaction function on the basis of our data sample, where the short-term interest rate is the German overnight money market rate.[33] Table 2 shows the results of two regressions. The first is based on equation (5.1). This standard Taylor rule provides a quite good fit for our sample period. The coefficients of the output gap and the inflation gap are significant, and the latter fulfills what is known as the Taylor-principle. The coefficient of the output gap is rather low but significant.

Table 2: Monetary Policy Rule Estimates

	Taylor Rule	**CGG** (q=12, l=3)
Constant	5.85	4.06
	(0.105)	(0.34)
g_p	1.19	1.68
	(0.071)	(0.132)
g_y	0.17	0.089
	(0.024)	(0.041)
r	–	0.89
		(0.017)
R^2	0.62	0.98

Standard errors in parentheses

[32] We also used a HP filter and the reported results were robust against this change. Furthermore, it turned out that a quadratic trend did not change our measure of the output gap, since the quadratic term turned out to be insignificant. Thus, we opted for the linear trend.

[33] In the term structure estimation the one month rate is the shortest maturity and we will interpret this as the policy rate. However, the correlation between the overnight money market rate and one month rate in our data set is 0.9951. The results in Table 2 are robust against a change in the dependent variable.

In order to estimate the forward-looking monetary policy rule in equation (5.2), we follow the usual procedure and implement a GMM estimation.[34] We depart from Clarida, Gali and Gertler (1998) and set $q = 12$ and $l = 3$, which yields a better fit compared to the specification without a forward-looking output gap. In the regression, we use the following set of instruments: a constant as well as lagged values of the output gap, the inflation gap, the IMF commodity price index and the short-term interest rate.[35] The results are presented in Table 2. In the forward-looking specification, the coefficient of the inflation gap is again significant and of a higher value compared to the plain Taylor rule specification. The coefficient of the output gap is of comparable size. The smoothing parameter is quite high but is in the expected range. In the next section we will use these results as a benchmark for the joint estimation of the monetary policy rule coefficients and the term structure dynamics.

6.3 A Joint Estimation of Monetary Policy Rule Coefficients and the Term Structure Dynamics

The estimation relies on a Kalman-filter-based maximum likelihood estimator.[36] This approach allows us to estimate the parameters of the model without directly observing the risk factors. However, we suspect that two of the factors are connected to an inflation and an output measure through eqs. (5.9) and (5.10). In order to apply a Kalman filter we first have to write the model in the linear state-space form, with the measurement equation resulting from the recursive restrictions and the identification of the factors, and the transition equation resulting from the assumed dynamics of the risk factors.

In particular, in the ‘3-factor partly-identified’ case, where two factors are linked to the inflation gap and the output gap, and the third factor is unobservable, the measurement equation is:

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[34] We correct for heteroscedasticity and autocorrelation of unknown form with a lag truncation parameter of 12. In addition, we chose Bartlett weights to ensure positive definiteness of our estimated variance-covariance matrix.

[35] More specifically, we use the first six, the ninth and the twelfth lag of the output gap, inflation gap and the IMF commodity price index as well as the first, sixth, ninth and twelfth lag of the short term interest rate. This is close to the instruments suggested by Clarida, Gali and Gertler (1998).

[36] The Kalman filter approach has gained popularity in the affine term structure literature. See, for example, Duan and Simonato (1995), Lund (1997), Gong and Remolona (1997), Geyer und Pichler (1998), Babbs and Nowman (1999), De Jong (2000), Bolder (2001) and Cassola and Luis (2003).

$$\begin{bmatrix} i_{1,t} \\ i_{3,t} \\ \vdots \\ i_{120,t} \\ \tilde{p}_t \\ \tilde{y}_t \end{bmatrix} = \begin{bmatrix} a_1 \\ a_3 \\ \vdots \\ a_{120} \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} b_{1,1} & b_{1,2} & b_{1,3} \\ b_{3,1} & b_{3,2} & b_{3,3} \\ \vdots & \vdots & \vdots \\ b_{120,1} & b_{120,2} & b_{120,3} \\ 1/(j_p)^q & 0 & 0 \\ 0 & 1/(j_y)^l & 0 \end{bmatrix} \cdot \begin{bmatrix} z_{p,t} \\ z_{y,t} \\ z_{3,t} \end{bmatrix} + \begin{bmatrix} n_{1,t} \\ n_{3,t} \\ \vdots \\ n_{120,t} \\ n_{p,t} \\ n_{y,t} \end{bmatrix}, \tag{6.1}$$

with $a_i = A_i / i$ and $b_{i,j} = B_{i,j} / i$, where the index i represents the number of months to maturity (i = 1, 3, 12, 24, 36, 48, 60, 84, 120) and j the index of the (up to three) risk factors. Expression (6.1) shows that we estimate a system of equations, i.e. a panel for different yields to maturity, with the additional interpretation that the first equation is the monetary policy reaction function of the central bank. Or, from a different angle, we can interpret our exercise as estimating a monetary policy rule with the additional restriction of the absence of arbitrage in financial markets. In shorter notation we might write the measurement system as:

$$Y_t = A + B \cdot Z_t + v_t \,. \tag{6.2}$$

The corresponding transition equation is represented by:

$$\begin{bmatrix} z_{p,t+1} \\ z_{y,t+1} \\ z_{3,t+1} \end{bmatrix} = \begin{bmatrix} j_p & 0 & 0 \\ 0 & j_y & 0 \\ 0 & 0 & j_3 \end{bmatrix} \cdot \begin{bmatrix} z_{p,t} \\ z_{y,t} \\ z_{3,t} \end{bmatrix} + \begin{bmatrix} s_p & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & s_3 \end{bmatrix} \cdot \begin{bmatrix} e_{p,t+1} \\ e_{y,t+1} \\ e_{3,t+1} \end{bmatrix} \tag{6.3}$$

or, in short:

$$Z_{t+1} = F \cdot Z_t + u_{t+1} \,. \tag{6.4}$$

In standard linear state-space models, no restrictions link the measurement and the transition equation. In this particular setup, however, the measurement equation comes from the transition equation and the no-arbitrage condition, that together yield the recursive restrictions on the factor loadings. Together with our identification of two risk factors the latter constitutes the measurement equation.

In the ‘2-factor identified’ case, the state-space system is written without the third factor leaving the identification for the two remaining factors unchanged. Additionally, we estimate a ‘2-factor non-identified’ case that resembles a traditional affine term structure model with only unobserved factors. The exact Kalman filter algorithm used in the paper is described in Appendix B.[37]

Table 3 presents the results of the three specifications. The first specification is the ‘2-factor non-identified’ case. The second is the 2-factor model with both factors being identified with the expected inflation and output gap respectively (‘2-factor identified’). Since the ‘2-factor identified’ specification might be very restrictive, we also estimated the ‘3-factor partly-identified’ specification in which two factors are identified as in the previous case and the third factor being unobserved. In both identified specifications we set q = 12 and l = 3 as we did in the benchmark estimation of the forward-looking specification of the monetary policy rule.[38]

Especially in the ‘3-factor partly-identified’ specification, the reaction coefficients are close to the levels that we obtained in the estimation of the forward-looking monetary policy rule before. This indicates that the estimated policy rule is consistent with the absence of arbitrage opportunities in financial markets. The reaction coefficient, the volatility and the coefficient of mean reversion of the third unobserved factor are also significant but its associated market price of risk is not.[39] This suggests that the risk premia in the German term structure of interest rates mainly stems from the inflation and the output risk. Note that only the inflation risk price is negative while the output risk price is positive.

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[37] The algorithm was performed by numerical optimization using Matlab codes. The starting values that are needed for the optimization were obtained by minimizing the sum of squared residuals between the estimated mean yields and the mean observed yields setting the value of the factors to zero (which also reflects their long-run mean by definition).

[38] We also estimated versions with no forward looking behavior (i.e. $q = l = 0$), but the fit was not as good as for the forward-looking specifications.

[39] If the third factor includes the smoothing behavior, this result seems quite natural because there is little risk associated with this factor.

Table 3: Parameter Estimates of the Gaussian Term Structure Model

	2F_non_ident	2F_ident	3F_ident
$j_1(j_y)$	0.87 (0.0025)	0.82 (0.0047)	0.98 (0.001)
$j_2(j_p)$	0.99 (0.0004)	0.98 (0.0003)	0.99 (0.002)
j_3	–	–	0.89 (0.0058)
$s_1(s_y)$	0.006 (0.0001)	0.0158 (0.0002)	0.00406 (0.0001)
$s_2(s_p)$	0.0016 (0.0001)	0.0007 (0.00001)	0.00045 (0.0001)
s_3	–	–	0.00286 (0.0001)
$l_1(l_y)$	-1.364 (0.2456)	7.19 (0.2317)	1.743 (1.55)
$l_2(l_p)$	-52.16 (0.6664)	-113.89 (0.8467)	- 139.39 (4.69)
l_3	–	–	-3.688 (4.81)
d	0.0038 (0.0001)	0.0043 (0.0001)	0.0043 (0.0001)
$g_1(g_y)$	1.363 (0.0287)	0.092 (0.0027)	0.088 (0.0153)
$g_2(g_p)$	0.934 (0.0240)	2.374 (0.0369)	1.764 (0.0469)
g_3	–	–	2.566 (0.116)

Standard errors in parentheses

The mean-reversion coefficients of inflation and output are quite high but close to the coefficients we obtained in a pure AR(1) regression of both. These values for the sample are 0.97 (0.91) for the inflation gap (output gap) with a R^2 of 0.93 (0.93). This suggests that our identification based on a simple AR(1) process is not too far from reality and seems consistent with the data.

Figure 2 shows the estimated dynamics of the term structure for the preferred '3-factor partly-identified' case. It reveals that the model captures the overall movement in the term structure that is shown in Figure 1(a) very well.

Figure 2: Estimated Term Structure Dynamics

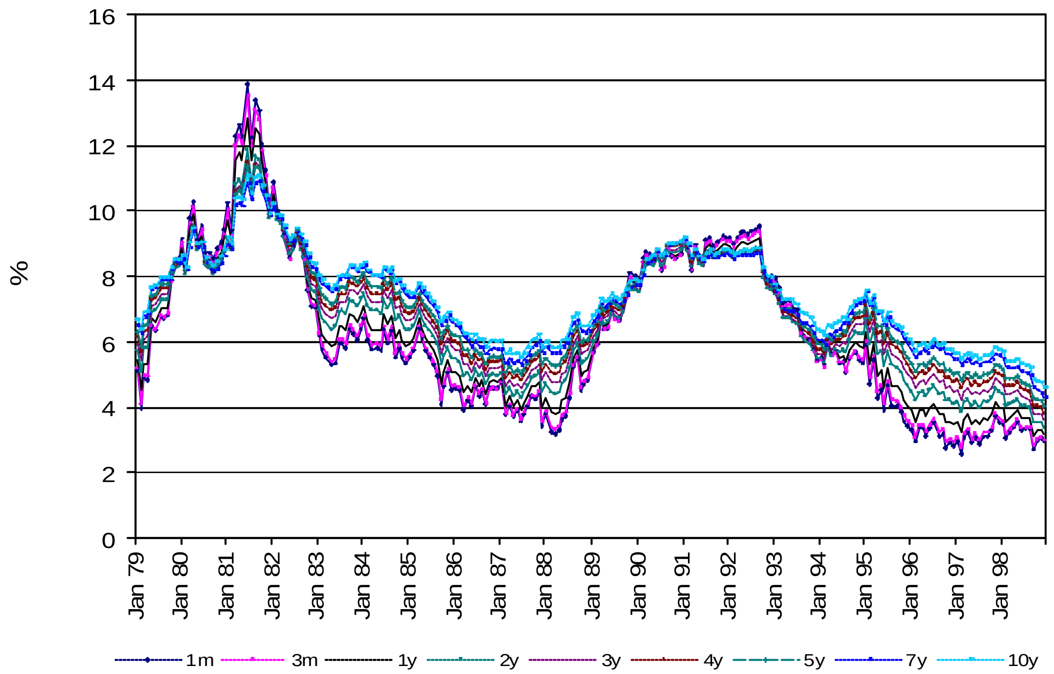


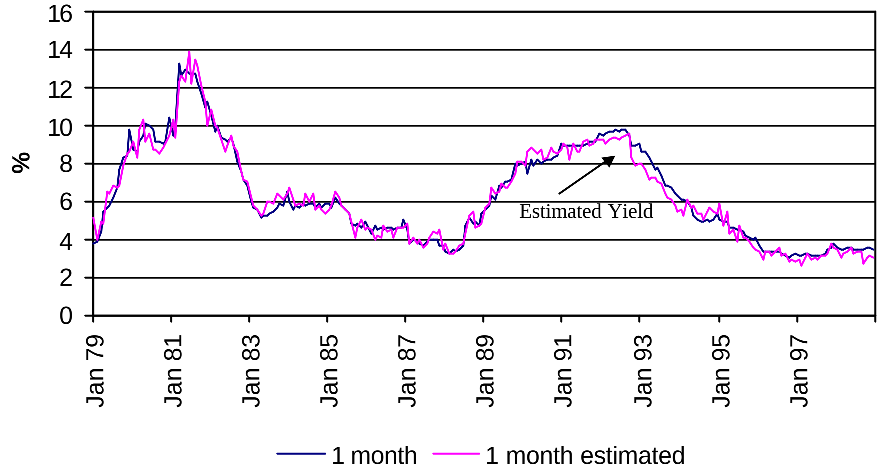
Figure 3 shows the estimated and observed yields for three particular maturities that represent the short end, the middle range and the long end of the yield curve. It shows that the model especially has a good fit at the middle range and the short end. This might not surprise since we model time-constant (i.e. only maturity dependent) risk premia.

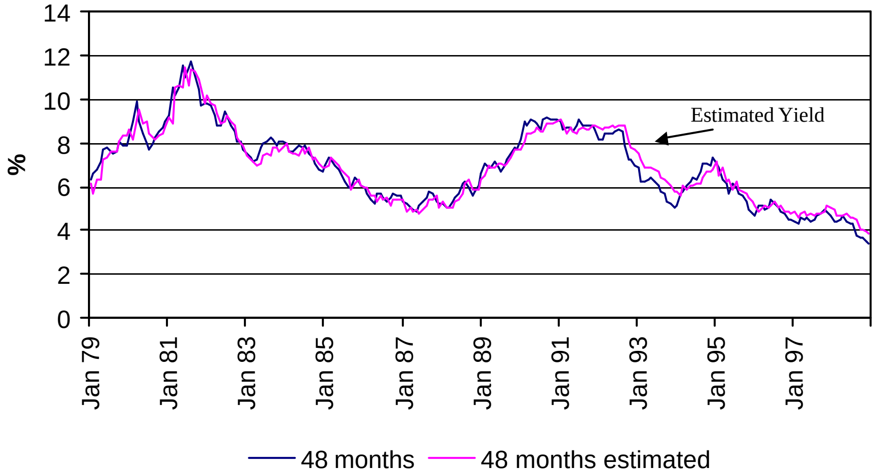
Figure 4 answers the question whether the model is able to capture the periods of an inverse term structure of interest rates. It shows the observed and the estimated spread defined as the yield of 10-year bonds minus the yield of one-month bonds. Again the overall fit is good.

Figure 5 presents the loadings that are associated with each factor in the '3-factor partly-identified' case. It shows that the 'inflation factor' has a nearly equal impact on all maturities. In a standard interpretation we would consider this as the 'level factor'. This interpretation seems quite reasonable given the high persistence of this factor as well as the fact that it incorporates the inflation target of the central bank, and thus the long-run expectations of inflation. Through the Fisher relation, this results in a long-run level of interest rates. The impact of the 'output factor' declines stronger along the yield curve. This is consistent with the interpretation as the 'curvature factor'. The third unobserved factor has its major impact on the short end of the yield curve. This is in line with the third 'smoothing factor' in monetary policy rules that incorporate interest rate smoothing behavior which has its mean impact on the short end of the yield curve. Independent of its precise interpretation, the third factor can be considered as the 'slope factor'.[40]

[40] We could also interpret this factor as capturing the forecast error that emerges in the error term of the forward looking specification of the rule.

Figure 3. Observed and Estimated Yields for Particular Maturities





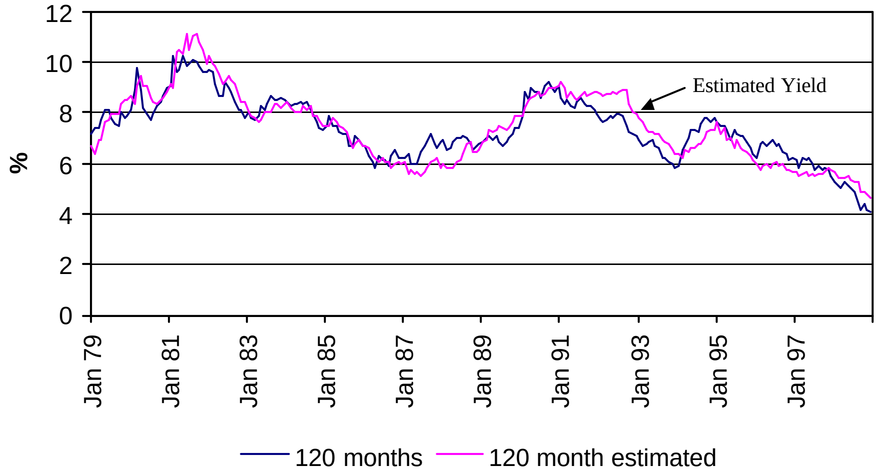


Figure 4: Observed and Estimated Spread (10y – 1m)

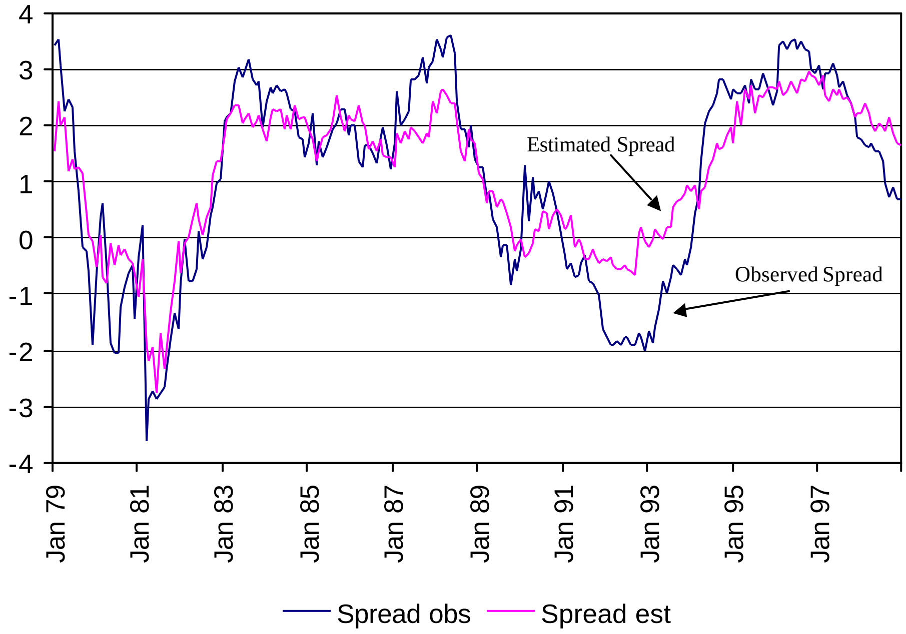
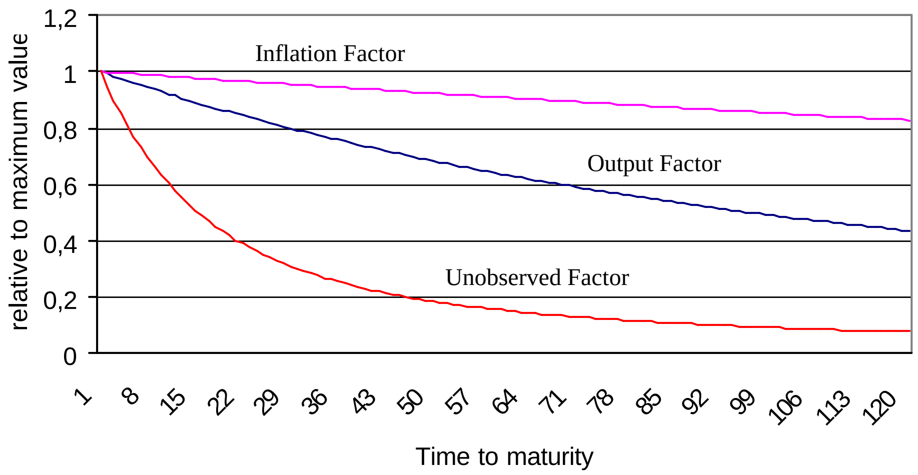
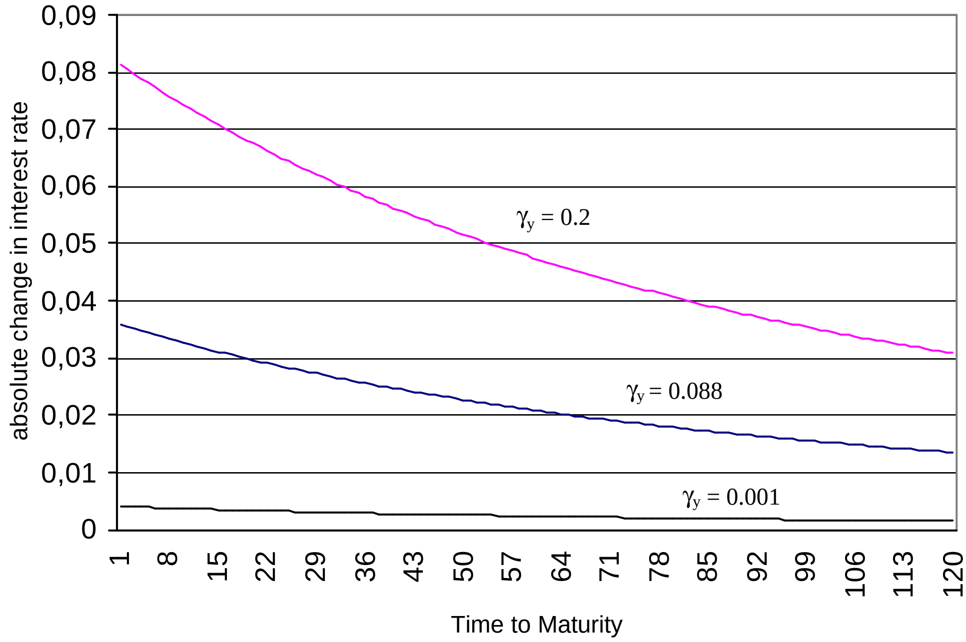


Figure 5: The Factor Loadings



From (4.5) we see that the monetary policy reaction coefficients also influence the factor loadings. However, they do not influence the relative impact of a shock to the particular risk factor along the yield curve as indicated in Figure 5. The strength of the policy reaction to a given shock rather determines the absolute impact of a shock on the yield curve. For instance, a stronger output response *ceteris paribus* leads to greater importance of the 'curvature factor' relative to the other factors, meaning that it is more likely that the yield curve changes its curvature. In this sense, monetary policy reaction determines to what extent given shocks alter the yield curve. Figure 6 shows the change in interest rates along the yield curve due to a shock to the output factor of the size of one standard deviation. The stronger the policy reaction to this given shock is, the greater is the absolute difference between the short rate and the long rate change, thus, the higher the effect on the curvature of the yield curve.

Figure 6: The Influence of the Policy Reaction



7 Conclusions

The paper constructed a Gaussian affine term structure model with a clear economic underpinning of the factors that drive the dynamics of the German term structure of interest rates. It shows that matching two out of three factors with the expected inflation gap and the expected output gap yields a reasonable empirical characterization of the German yield curve between 1979 and 1998. Furthermore, the empirical results are consistent with the results obtained from a forward-looking monetary policy reaction function that only cares about the dynamics of the short-term interest rate, or the policy rate. However, the monetary policy reaction function that is extracted from the affine term structure model has the additional feature that it is consistent with the absence of arbitrage in financial markets.

Still, there are some avenues for future research to improve our analysis. First, the possibility of time-varying term premia should improve our results. This is commonly modeled by defining the risk premia as a linear function of the risk factors.[41] Second, an improvement of the identification of the risk factors is preferable, since the AR(1) process for the inflation and the output gap, though being a good first approximation, seems too restrictive. Third, other macroeconomic factors such as foreign interest rates, monetary aggregates or the exchange rate might as well influence the dynamics of the term structure of interest rates. Future research should aim to account for these additional influences.

[41] Another possibility is to specify the risk premia as a multiple of the volatility of the underlying shocks, as this is the case in the general DK-framework (see equation (3.9)). However, the alternative specification for the compensation of interest rate risk to vary independently of such volatility has proved useful especially in forecasting future bond yields (see Duffee, 2002).

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Appendix A.1: Recursive Restrictions

The derivation starts with writing the expression (2.1) in logs (for convenience we let the bond in time *t* mature in *n*+1 periods):

$$p_{n+1,t} = \log\left(E\left[M_{t+1}P_{n,t+1}\right]\right). \tag{A.1}$$

With the assumption of joint log-normality of bond prices and the nominal pricing kernel and using the statistical principle that if $\log X \sim N(m, s^2)$ then $\log E(X) = m + \frac{s^2}{2}$, it follows that:

$$p_{n+1,t} = E_t\left[m_{t+1} + p_{n,t+1}\right] + \frac{1}{2}Var_t\left[m_{t+1} + p_{n,t+1}\right]. \tag{A.2}$$

From (3.2) and (3.4) we get

$$m_{t+1} + p_{n,t+1} = -\left(d + g'Z_t + l\,V(Z_t)^{1/2}e_{t+1}\right) - A_n - B_n'Z_{t+1}. \tag{A.3}$$

We can substitute for Z_{t+1} from equation (3.1) and after sorting terms we arrive at the following expression:

$$\begin{aligned} m_{t+1} + p_{n,t+1} = &-\left(A_n\right) + d + B_n'(1-\Phi)\cdot q - \left(g' + B_n'\Phi\right)\cdot Z_t \\ &- \left(l' + B_n'\right)\cdot V(Z_t)^{1/2}e_{t+1}, \end{aligned} \tag{A.4}$$

so that

$$E_t\left[m_{t+1} + p_{n,t+1}\right] = -\left(A_n\right) + d + B_n'(1-\Phi)\cdot q - \left(g' + B_n'\Phi\right)\cdot Z_t, \tag{A.5}$$

$$Var_t\left[m_{t+1} + p_{n,t+1}\right] = \left(l' + B_n'\right)^2\cdot V(Z_t). \tag{A.6}$$

Substituting these moments into (A.2) and recognizing the typical diagonal elements in the variance-covariance matrix we have after sorting the terms:

$$-p_{n+1,t} = A_n + d + B_n'(1-\Phi)\cdot q - \frac{1}{2}\left(\sum_{j=1}^{k}\left(l_j + B_{j,n}\right)^2 \cdot a_j\right)$$
$$+\left[g' + B_n'\Phi - \frac{1}{2}\sum_{j=1}^{k}\left(l_j + B_{j,n}\right)^2 \cdot b_j'\right]\cdot Z_t. \quad \text{(A.7)}$$

Matching the variables in order to verify that

$$-p_{n+1,t} = A_{n+1} + B_{n+1}' \cdot Z_t \quad \text{(A.8)}$$

it must be that

$$A_{n+1} = A_n + d + B_n'(I-\Phi)\cdot q - \frac{1}{2}\sum_{j=1}^{k}\left(l_j + B_{j,n}\right)^2 \cdot a_j \quad \text{(A.9)}$$

$$B_{n+1}' = \left(g' + B_n'\Phi\right) - \frac{1}{2}\sum_{j=1}^{k}\left(l_j + B_{j,n}\right)^2 \cdot b_j' \quad . \quad \text{(A.10)}$$

The last two expressions correspond to the recursive restrictions stated in the text. The recursive restrictions of the Gaussian version in section 4 of the paper follow by the same steps, or can be directly obtained by substituting the parameter restrictions (4.1) into the just derived general restrictions.

Appendix A.2: Derivation of the Term Premia

The term premia is defined as the difference between the 'holding period return' and the one-period interest rate:

$$\Lambda_{n,t} = E_t p_{n,t+1} - p_{n+1,t} - i_{1,t}. \quad \text{(A.11)}$$

The individual elements of (A.11) are:

$$E_t p_{n,t+1} = -A_n - B_n' \cdot E_t\left[Z_{t+1}\right] = -A_n - B_n'(I-\Phi)\cdot q - B_n'\Phi \cdot Z_t \quad \text{(A.12)}$$

$-p_{n+1,t} = A_{n+1} + B_{n+1}' \cdot Z_t$ and (A.13)

$$i_{1,t} = d - \frac{1}{2}\sum_{j=1}^{k} l_j^2 \cdot a_j + \left(g' - \frac{1}{2}\sum_{j=1}^{k} l_j^2 \cdot b_j' \right) \cdot Z_t \qquad \text{(A.14)}$$

Substituting the recursive restrictions (3.6) and (3.7) into (A.13) and combining the elements according to the definition (A.11) yields after sorting the terms we get:

$$\Lambda_{n,t} = -\frac{1}{2}\sum_{j=1}^{k}(l_j + B_{j,n})^2 \cdot a_j - \left[\frac{1}{2}\sum_{j=1}^{k}(l_j + B_{j,n})^2 \cdot b_j'\right] \cdot Z_t + \frac{1}{2}\sum_{j=1}^{k} l_j^2 a_j + \left[\frac{1}{2}\sum_{j=1}^{k} l_j^2 b_j'\right] \cdot Z_t$$

Solving the binomials and sorting terms leads to the expression of the term premia in the text:

$$\Lambda_{n,t} = -\sum_{j=1}^{k}\left(l_j B_{j,n} + \frac{B_{j,n}^2}{2} \right) \cdot a_j - \sum_{j=1}^{k}\left(l_j B_{j,n} + \frac{B_{j,n}^2}{2} \right) \cdot b_j' \cdot Z_t , \qquad \text{(A.15)}$$

which is equivalent to (3.9) in the text given the particular structure of the variance-covariance matrix.

Appendix B: The Kalman Filter

The Kalman Filter recursively computes the optimal estimate of a state variable at the moment t based on the information available up to period t-1. Unknown parameters are usually estimated by a maximum likelihood procedure. This appendix describes how the Kalman Filter is used in the paper.[42] In order to perform the Kalman Filter the model has to be written in state space form, as in equations (6.2) and (6.4). The first equation that builds on the recursive restrictions is the observation or measurement equation that has the following general form:

$$Y_t = A + B \cdot Z_t + v_t \qquad \text{(B.1)}$$

where $v_t \sim N(0, R)$.

The state or transition equation has the following form:

[42] The Kalman Filter and the maximum likelihood estimations were carried out using Matlab codes. Details on the Kalman Filter can be found in Hamilton (1994, ch. 13).

$$Z_{t+1} = C + F \cdot Z_t + u_{t+1} \tag{B.2}$$

where $u_{t+1} \sim N(0, Q)$.

The Kalman Filter procedure generally consists of five steps, that are described next:

1. Initializing the state vector

The first step of the Kalman Filter is the initializing of the state vector Z_t at t=1 to start the recursion. $\hat{Z}_0$ can be seen as a guess concerning the value of Z using all information up to t=0. P_0 is the uncertainty about the prior guess. Using $\hat{Z}_0$ and P_0 in (B.2), the optimal estimator for Z_1 is given by:

$$\hat{Z}_{1|0} = C + F \cdot \hat{Z}_0 . \tag{B.3}$$

The associated mean squared error (MSE) matrix is

$$\begin{aligned} P_{1|0} &\equiv E\left[(Z_1 - \hat{Z}_{1|0})(Z_1 - \hat{Z}_{1|0})'\right] \\ &= FP_{1|0}F' + Q. \end{aligned} \tag{B.4}$$

If the eigenvalues of F are all inside the unit circle, the process for Z_t in (B.2) is covariance-stationary. In this case the Kalman Filter iteration can be started with $\hat{Z}_{1|0}$ = 0 and $P_{1|0}$ whose elements expressed as a column vector are (Hamilton, 1994, p. 378):

$$vec(P_{1|0}) = \left[I_{k^2} - (F \otimes F)\right]^{-1} \cdot vec(Q),$$

with k being the number of unobservable factors.

2. Forecasting the measurement vector $\mathbf{Y_t}$

Given the starting values $\hat{Z}_{1|0}$ and $P_{1|0}$ the next step is to calculate the respective values for the following dates. Generalizing the equations (B.3) and (B.4), we have the following prediction equations:

$$\hat{Z}_{t|t-1} = C + F \cdot \hat{Z}_{t-1} \text{ and} \tag{B.5}$$

$$P_{t|t-1} = FP_{t|t-1}F' + Q. \tag{B.6}$$

The forecast of Y_t can, thus, directly be obtained from (B.1):

$$\hat{Y}_{t|t-1} = A + B \cdot \hat{Z}_{t|t-1} \tag{B.7}$$

with the following forecast MSE:

$$E\left[(Y_t - \hat{Y}_{t|t-1})(Y_t - \hat{Y}_{t|t-1})'\right] = BP_{t|t-1}B' + R. \tag{B.8}$$

3. Updating the inference about the state vector

Since Z_t and Y_t are related by the measurement equation, the knowledge of the value Y_t can be used to update $\hat{Z}_{t|t-1}$. In order to perform this, we need to characterize the joint distribution of the observation and the state vector. Therefore, we need to compute the conditional covariance between the two forecast errors:

$$E\left[(Y_t - \hat{Y}_{t|t-1})(Z_t - \hat{Z}_{t|t-1})'\right] = BP_{t|t-1}. \tag{B.9}$$

Using equations (B.7), (B.8) and (B.9), the conditional distribution of the vector (Y_t, Z_t) given the information I_{t-1} is:

$$\begin{bmatrix} Y_t | I_{t-1} \\ Z_t | I_{t-1} \end{bmatrix} \sim N\left(\begin{bmatrix} A + B\hat{Z}_{t|t-1} \\ \hat{Z}_{t|t-1} \end{bmatrix}, \begin{bmatrix} BP_{t|t-1}B' + R & BP_{t|t-1} \\ P_{t|t-1}B' & P_{t|t-1} \end{bmatrix} \right). \tag{B.10}$$

From (B. 10) we obtain the following updating equations:[43]

[43] We made use of the following general rule: if two variables X_1 and X_2 have a joint normal conditional distribution given by $\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} \sim N\left(\begin{bmatrix} m_1 \\ m_2 \end{bmatrix}, \begin{bmatrix} \Delta_{11} & \Delta_{12} \\ \Delta_{21} & \Delta_{22} \end{bmatrix} \right)$, then the distribution of $X_2 | X_1$ is $N \sim (m, \Sigma)$, with $m = m_2 + \Delta_{21}\Delta_{11}^{-1}(X_1 - m_1)$ and $\Sigma = \Delta_{22} - \Delta_{21}\Delta_{11}^{-1}\Delta_{12}$.

$$\hat{Z}_{t|t} = \hat{Z}_{t|t-1} + P_{t|t-1}B'(BP_{t|t-1}B' + R)^{-1}\left[Y_t - (A + B\hat{Z}_{t|t-1})\right] \text{ and} \tag{B.11}$$

$$P_{t|t} = P_{t|t-1} - P_{t|t-1}B'(BP_{t|t-1}B' + R)^{-1} BP_{t|t-1}. \tag{B.12}$$

4. Forecasting the state vector

After estimating $\hat{Z}_{t|t}$ and $P_{t|t}$, we can proceed with the estimation of $\hat{Z}_{t|t+1}$ and $P_{t|t+1}$. Using equations (B.5) and (B.6), we get:

$$\hat{Z}_{t+1|t} = C + F\hat{Z}_{t|t} \quad \text{and} \tag{B.13}$$

$$P_{t+1|t} = FP_{t|t}F' + Q. \tag{B.14}$$

5. Maximum likelihood estimation of parameters

The log-likelihood function is built up recursively

$$\log L(\tilde{Y}_T) = \sum_{t=1}^{T} \log\ f(Y_t \left| I_{t-1}\right.), \tag{B.15}$$

where

$$f(Y_t \left| I_{t-1}\right.) = (2p)^{-n/2}\left|BP_{t|t-1}B' + R\right|^{-1/2} \cdot \exp\left\{-\frac{1}{2}(Y_t - A - B\hat{Z}_{t|t-1})'(BP_{t|t-1}B' + R)^{-1}(Y_t - A - B\hat{Z}_{t|t-1})\right\}$$

where n is here the dimension of the vector Y. The parameters of the measurement and the state equations can then be obtained by a numerical maximization of the likelihood function.[44] Finally, the standard errors can be numerically computed on estimates of the inverse of the Hessian matrix at the convergence point. See Judd (1998) for the so-called ‘finite difference method’.

[44] In the Matlab program we depart from (B.15) and compute the negative of the log-likelihood function. The optimization procedure is then performed as a minimization problem instead. Furthermore we abstracted from the constant in the likelihood function because it only has a scaling function.

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