Stability and identification with optimal macro-prudential policy rules

JB Chatelain and K Ralf

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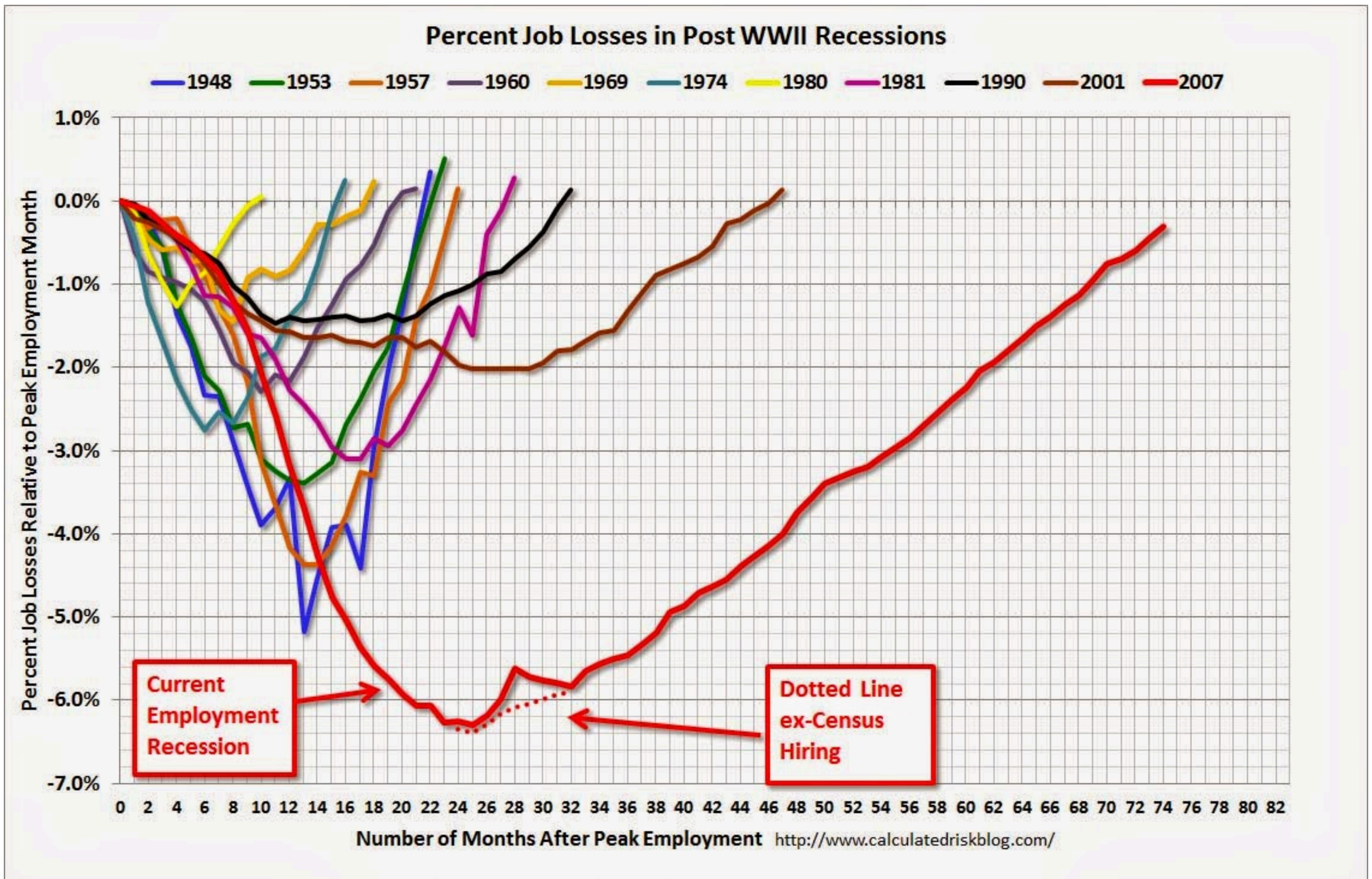
Financial stability concerns influence monetary policy (Stein 2014)

1. Quadratic loss objective which includes a risk term: variance of realized employment which depend on financial market vulnerability
2. Some variable summarizing financial market vulnerability is influenced by monetary policy.
3. Risks associated with FMV cannot be fully offset at zero cost with other non-monetary tools, such as financial regulation

The crisis and the loss function of the FED: (USA, beginning April 2014)



Macro-prudential DSGE and New Keynesian model concerns upon

1) Identification and ability to **test** those models and to then inform monetary policy:

Cochrane (2011), J Political Economy.

Komunjer and Ng (2011), Econometrica.

2) Lack of optimal control robustness to misspecification: local instability in m dimensions,

in order to achieve the unique solution for the model (determinacy) with expectations exact immediate self adjustment to shock in those *m* dimensions, for non pre-determined variables.

Plan

1. Blanchard Kahn (1980) unique solution
2. Kalman's (1960) Controllability
3. Quasi-optimal rules
4. Over stable optimal rules
5. Optimal rules robust to misspecification

1. Blanchard and Kahn (1980) no bubbles hypothesis

Blanchard and Kahn (1980) hypothesis for a unique solution

N pre-determined variables:

autoregressive shocks, capital stock

M non pre-determined variables:

output gap, inflation, asset prices, credit.

Expectations driven variables, so that even you observe the data now, next second they could be driven by sunspots shocks initial conditions.

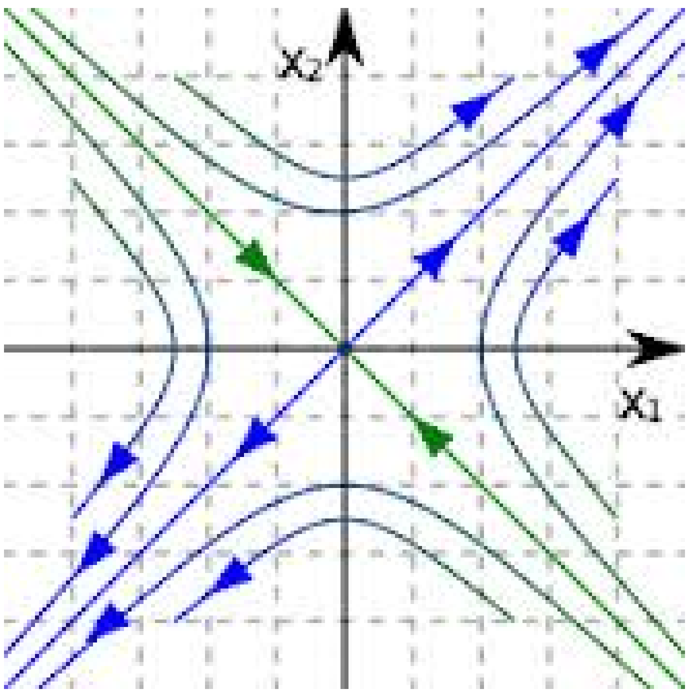
« Unique stable » solution when M unstable dimensions (exploding variables except on N stable dimensions),

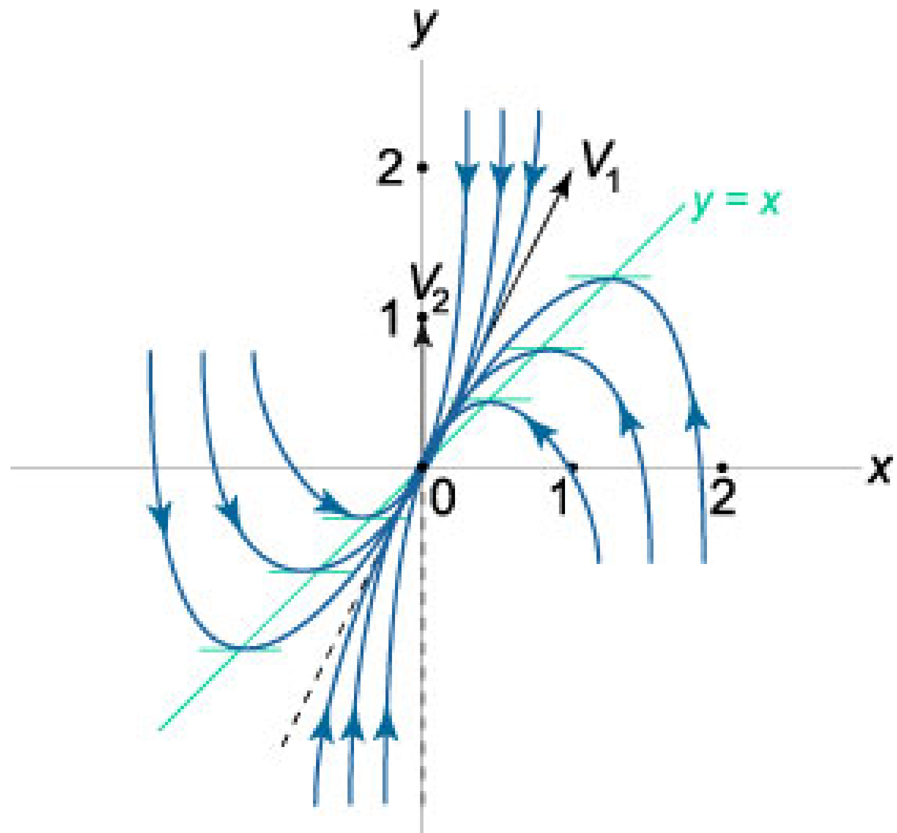
M non pre-determined are « determined » by N pre-determined, immediate self-correction to shocks in M dimension to remain on the stable manifold. The expectations of errors in (1b) is ALWAYS ZERO.

$$\text{(1a)} \qquad \underbrace{X_{t+1}}_{n_X \times 1} = \underbrace{A(\theta)}_{n_X \times n_X} \underbrace{X_t}_{n_X \times 1} + \underbrace{B(\theta)}_{n_X \times n_\varepsilon} \underbrace{\varepsilon_{t+1}}_{n_\varepsilon \times 1},$$

$$\text{(1b)} \qquad \underbrace{Y_{t+1}}_{n_Y \times 1} = \underbrace{C(\theta)}_{n_Y \times n_X} \underbrace{X_t}_{n_X \times 1} + \underbrace{D(\theta)}_{n_Y \times n_\varepsilon} \underbrace{\varepsilon_{t+1}}_{n_\varepsilon \times 1}.$$

2 dimensions linear systems





Blanchard and Kahn (1980) hypothesis w=(k,q,z)

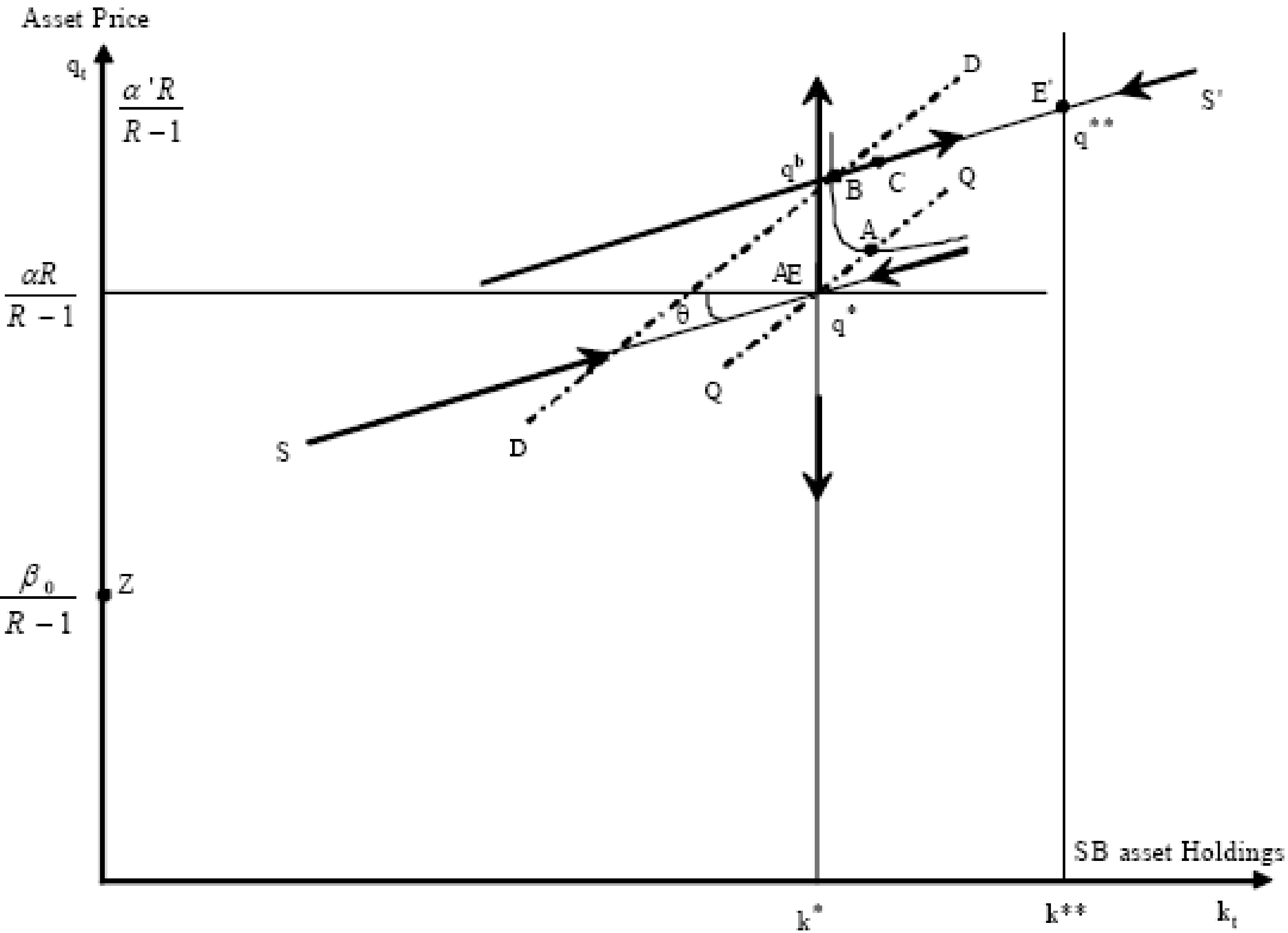
$$\forall t \in \mathbb{N}, \exists \overline{\mathbf{w}}_t \in \mathbb{R}^k, \exists \theta_t \in \mathbb{R}, \text{ such that}$$

$$\left| E_t \left(\mathbf{w}_{t+1} \mid \Omega_t \right) \right| \leq \left(1+i\right)^{\theta_t} \overline{\mathbf{w}}_t, \ \forall i \in \mathbb{R}^+.$$

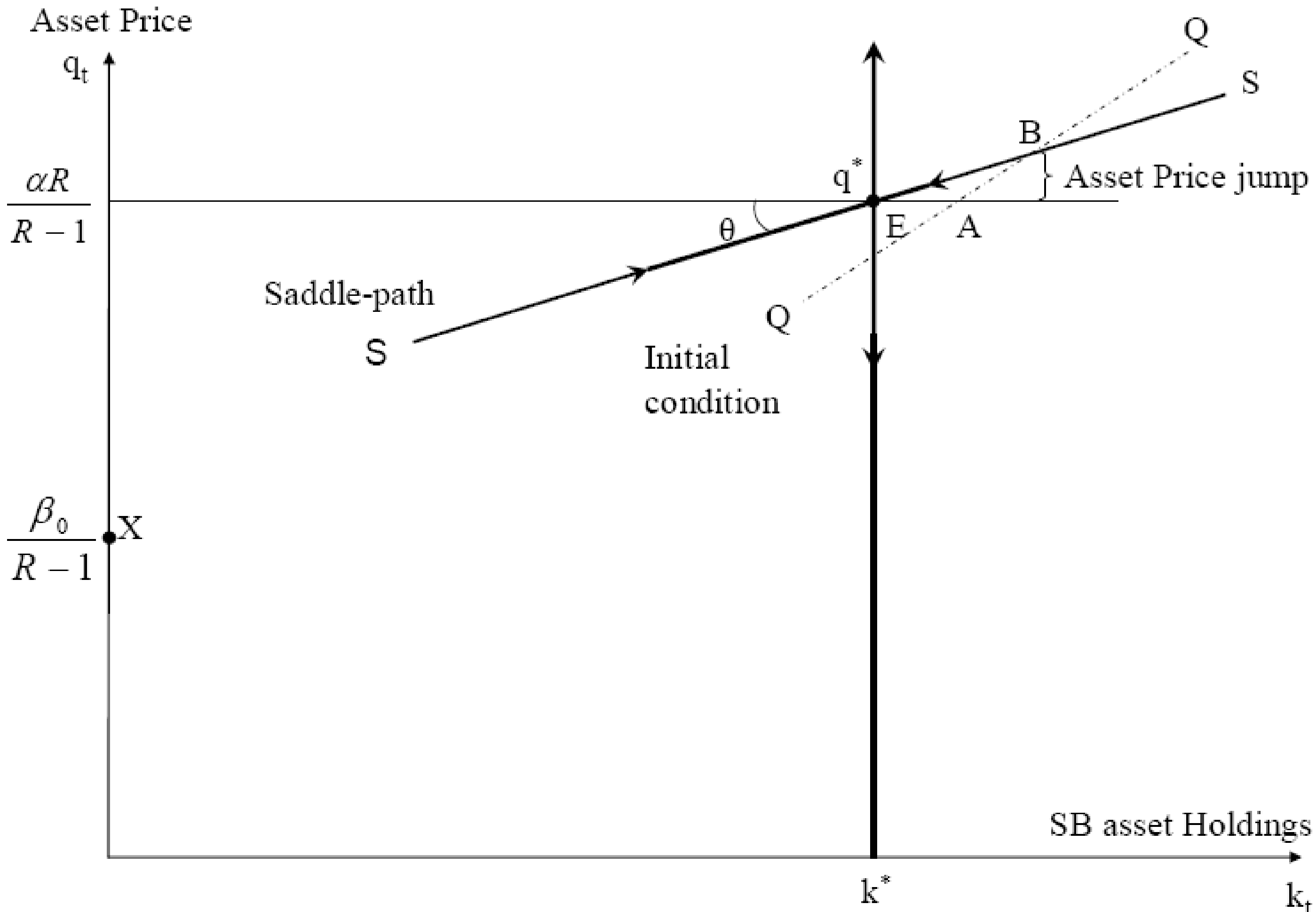
Excluding diverging path « bubbles » by assumption and not by an explicit stabilizing mechanism, for any R>1

$$\lim_{s\to+\infty} E_t\left(\frac{q_{t+s}}{\prod_{\tau=0}^{\tau=s} R_\tau}\right) = 0$$

Path AB: Divergent asset price alpha=A



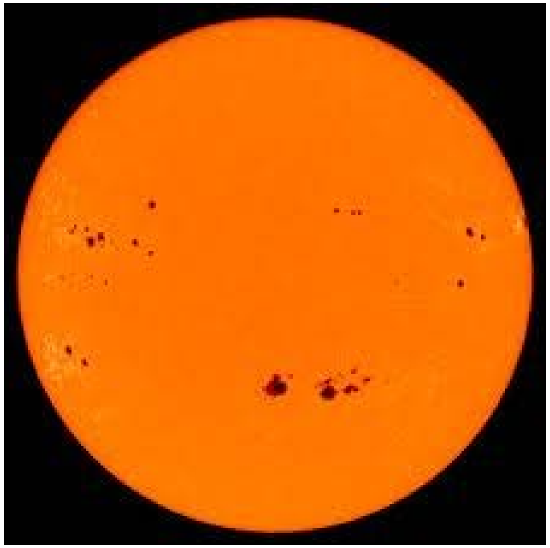
Asset price jump



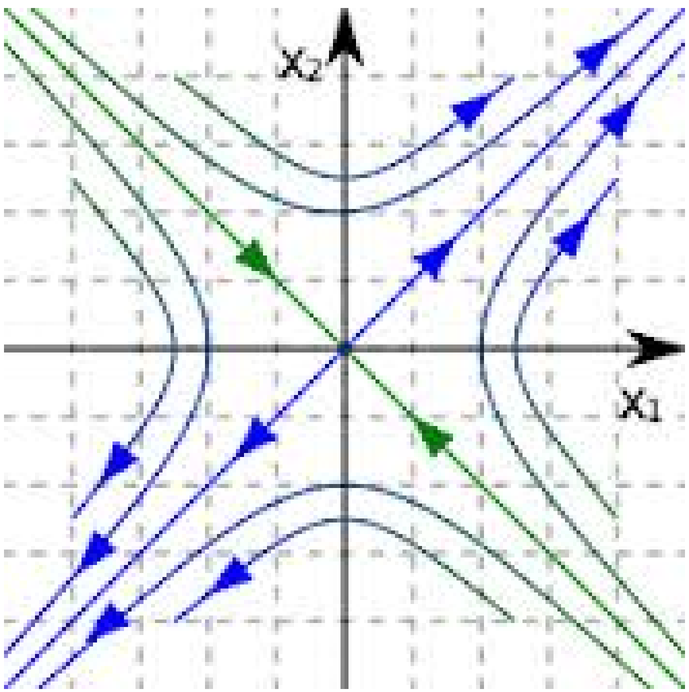
New Keynesian Central Bank rules should maintain *M* potential bubbles in the economy, else sunspots (infinity of initial conditions in *M* dimensions) are **worse**.

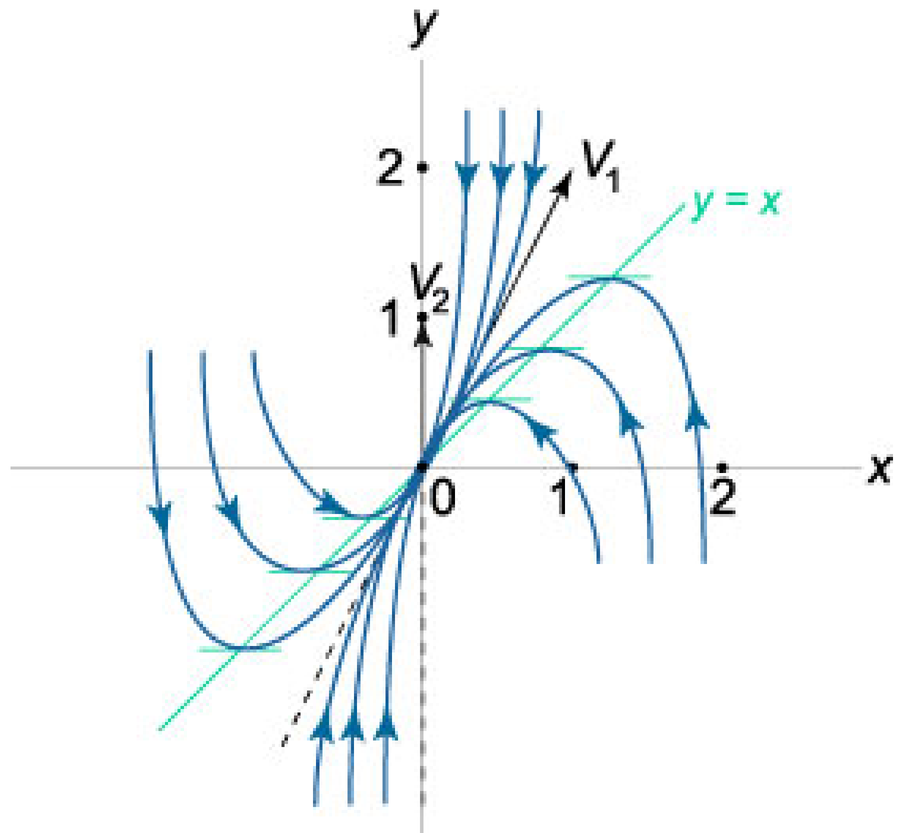
Paradise

Hell



« Determinacy »: Do not « over-stabilize » more than N dimensions (N<N+M)





Bubbles versus sunspot for modelling financial stability hard to communicate to

Governors of central banks

Other policy makers, journalists

Microeconomists, labour and finance economists (Cochrane)

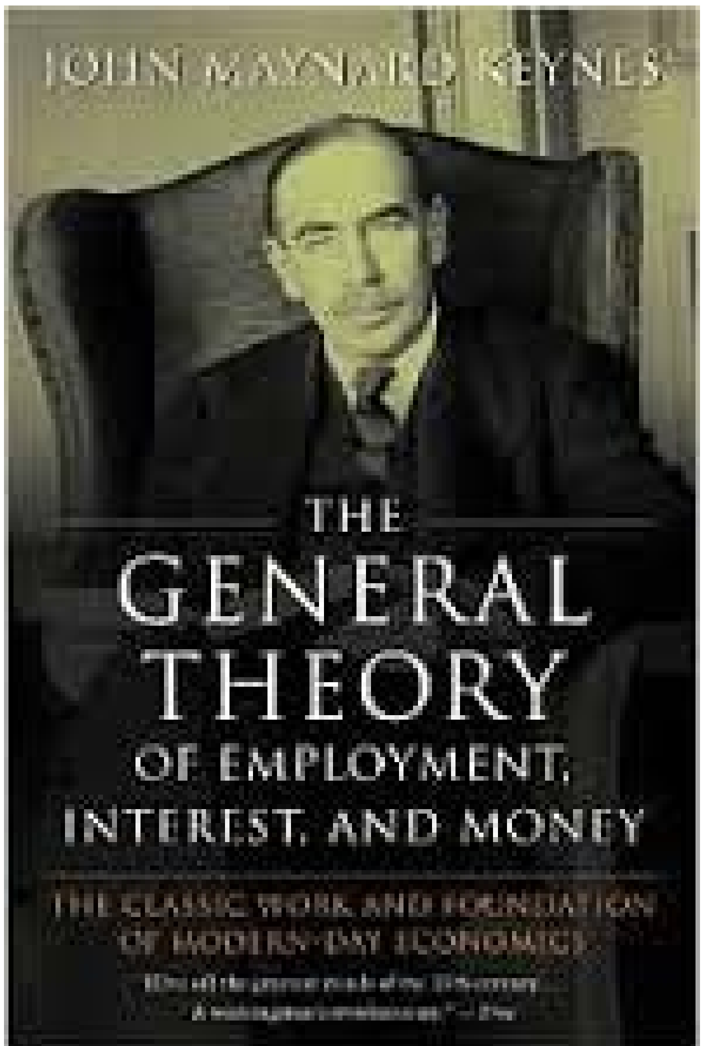
Mathematicians, engineers

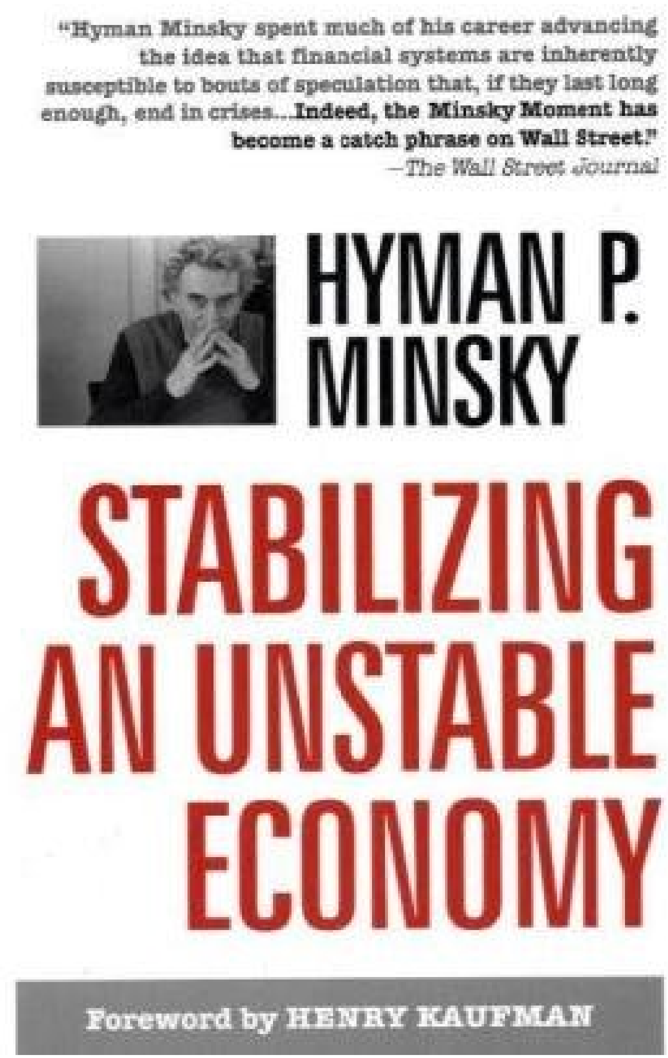
Businessmen

Households, poor people:

People expects « financial stability » means « lean against and stabilize bubbles ».

« Prefer potential bubbles instead of sunspots »: not their idea.





2. Kalman’s controllability (1960)

Kalman's controllability (1960)

Ability to control a dynamical linear system from point A to point B at any speed with a linear rule.

Effect of monetary policy on financial market vulnerability (Stein 2014).

No: if your instruments for control are not able to change several dimensions:

exogenous auto-regressive shocks.

Controllability Normal Form

Theorem 15 *There exists a state coordinate change which forms the linear system $\dot{x} = Ax + Bu$ into*

$$\begin{pmatrix} \dot{z}_1 \\ \dot{z}_2 \end{pmatrix} = \begin{pmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} + \begin{pmatrix} B_1 \\ 0 \end{pmatrix} u$$

such that (A_{11}, B_1) is controllable. In Matlab use `ctrbf`.

You should learn to read these equations actually as

$$\dot{z}_1 = A_{11}z_1 + A_{12}z_2 + B_1u, \quad \dot{z}_2 = A_{22}z_2.$$

Hence the evolution of $z_2(t)$ **cannot** be influenced by the contro

Definition 16 *The eigenvalues of A_{22} are called* **uncontrol** **modes** *of (A, B).*

Summary

Every system $\dot{x} = Ax + Bu$ can be transformed by state-coordinate change into the controllability normal form:

$$\begin{pmatrix} \dot{z}_1 \\ \dot{z}_2 \end{pmatrix} = \begin{pmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} + \begin{pmatrix} B_1 \\ 0 \end{pmatrix} u, \quad (A_{11}, B_1) \text{ controllable.}$$

- Controllability of (A_{11}, B_1) means that $\begin{pmatrix} B_1 & A_{11}B_1 & \cdots & A_{11}^{n-1}B_1 \end{pmatrix}$ has full row rank.

  Equivalently, $\begin{pmatrix} A_{11} - \lambda I & B_1 \end{pmatrix}$ has full row rank for all $\lambda \in \mathbb{C}$.

- The matrix $\begin{pmatrix} A - \lambda I & B \end{pmatrix}$ looses rank at $\lambda \in \mathbb{C}$ iff $\lambda \in \text{eig}(A_{22})$, i.e., exactly in the uncontrollable modes of (A, B).

- The evolution of the state z_2 cannot be influence by the control input.

  Intuitively, the uncontrollable modes of the system cannot be excited by control.

(1) Stabilizing **M** unstable dimensions

$X(t+1) = a \cdot X(t) + \mathbf{b} r(t), \quad r(t) = \mathbf{f} \cdot X(t)$

$X(t+1) = (a+\mathbf{bf}) \cdot X(t)$

IF b≠0 (Kalman controllability; scalar case);

$b = dX(t+1)/dr(t) < 0$, $f = dr(t)/dX(t) > 0$ OR

$b = dX(t+1)/dr(t) > 0$, $f = dr(t)/dX(t) < 0$

$$0 < a+\mathbf{bf} < 1 < a$$

(2) Stabilizing « more » **N** stable dimensions Increases the speed of convergence to steady state

$X(t+1)= a \, . \, X(t) + \mathbf{b}r(t), \quad r(t)= \mathbf{f}.X(t)$

$X(t+1)= (a+\mathbf{bf}\,).\, X(t)$

IF $b\neq 0$ (Kalman controllability; scalar case);

$b=dX(t+1)/dr(t)<0$, $f=dr(t)/dX(t)>0$ OR

$b=dX(t+1)/dr(t)>0$, $f=dr(t)/dX(t)<0$

$$0 < a+\mathbf{bf} < a < 1$$

3. Quasi-optimal rules

Controllability and the linear quadratic regulator

Minimize a quadratic loss function including a cost for changing the policy rate

Subject to a linear system, with linear policy rule.

Kalman controllability: can choose rules with as many stable dimensions as you wish (indeed all stable except in Blanchard Kahn world).

Unique solution for the fully stable parameters of the rule (N+M dimensions) and for ALL N+M stable eigenvalues > M required in Blanchard Kahn.

Controllability, Rules and Blanchard Kahn

Monetary policy rules can fully stabilize the system, but « indeterminacy for ad hoc rules».

Design your policy rules so that they leave M unstable dimensions, [although they are able to fully stabilize the model].

Financial stability?

Quadratic loss

$$\max_{\{r_t, k_{t+1}, q_{t+1}\}} -\frac{1}{2}\sum_{t=0}^{+\infty}\beta^t \left[\begin{array}{c} \left(\frac{\mathbf{k}_t-\mathbf{k}^*}{\mathbf{k}^*}\right)^T \mathbf{Q}_{nn}\left(\frac{\mathbf{k}_t-\mathbf{k}^*}{\mathbf{k}^*}\right) + \left(\frac{\mathbf{q}_t-\mathbf{q}^*}{\mathbf{q}^*}\right)^T \mathbf{Q}_{mm}\left(\frac{\mathbf{q}_t-\mathbf{q}^*}{\mathbf{q}^*}\right) \\ +\left(\frac{\mathbf{k}_t-\mathbf{k}^*}{\mathbf{k}^*}\right)^T \mathbf{Q}_{nm}\left(\frac{\mathbf{q}_t-\mathbf{q}^*}{\mathbf{q}^*}\right) + \left(\frac{\mathbf{q}_t-\mathbf{q}^*}{\mathbf{q}^*}\right)^T \mathbf{Q}_{mn}\left(\frac{\mathbf{k}_t-\mathbf{k}^*}{\mathbf{k}^*}\right) \\ +\rho\left(r_t - r^*\right)^2 \end{array}\right]$$

$$= -\left(\begin{array}{c} \frac{\mathbf{k}_0-\mathbf{k}^*}{\mathbf{k}^*} \\ \frac{\mathbf{q}_0-\mathbf{q}^*}{\mathbf{q}^*} \end{array}\right)^T \mathbf{P}\left(\begin{array}{c} \frac{\mathbf{k}_0-\mathbf{k}^*}{\mathbf{k}^*} \\ \frac{\mathbf{q}_0-\mathbf{q}^*}{\mathbf{q}^*} \end{array}\right) \qquad (1)$$

F are rules parameters
B are effets of controls upon state
z(t) are exogenous variables

$$\begin{pmatrix} \mathbf{k}_{t+1} \\ {}_t\mathbf{q}_{t+1} \end{pmatrix} = \left(\underbrace{\begin{pmatrix} \mathbf{A}_{nn} & \mathbf{A}_{nm} \\ \mathbf{A}_{mn} & \mathbf{A}_{mm} \end{pmatrix}}_{\mathbf{A}} + \underbrace{\begin{pmatrix} \mathbf{B}_{n1} \\ \mathbf{B}_{m1} \end{pmatrix}}_{\mathbf{B}} \underbrace{\begin{pmatrix} \mathbf{F}_{1n} & \mathbf{F}_{1m} \end{pmatrix}}_{-\mathbf{F}} \right) \begin{pmatrix} \mathbf{k}_t \\ \mathbf{q}_t \end{pmatrix} + \gamma \mathbf{z}_t$$

C=–N(F(s))
Depends upon F(s), the rule
s=n, stable dimensions

$$\begin{pmatrix} \mathbf{M}(\mathbf{F})_{nn} & \mathbf{M}(\mathbf{F})_{nm} \\ \mathbf{M}(\mathbf{F})_{mn} & \mathbf{M}(\mathbf{F})_{mm} \end{pmatrix} \begin{pmatrix} \mathbf{A}_{nn} - \mathbf{B}_{n1}\mathbf{F}_{1n} & \mathbf{A}_{nm} - \mathbf{B}_{n1}\mathbf{F}_{1m} \\ \mathbf{A}_{mn} - \mathbf{B}_{m1}\mathbf{F}_{1n} & \mathbf{A}_{mm} - \mathbf{B}_{m1}\mathbf{F}_{1m} \end{pmatrix}$$

$$= \begin{pmatrix} \mathbf{\Lambda}_{nn} & \mathbf{0}_{nm} \\ \mathbf{0}_{mn} & \mathbf{\Lambda}_{mm} \end{pmatrix} \begin{pmatrix} \mathbf{M}(\mathbf{F})_{nn} & \mathbf{M}(\mathbf{F})_{nm} \\ \mathbf{M}(\mathbf{F})_{mn} & \mathbf{M}(\mathbf{F})_{mm} \end{pmatrix}$$

$$E_t\mathbf{q}_{t+1} = -\mathbf{N}(\mathbf{F})_{mn}\mathbf{k}_{t+1} = -\mathbf{M}(\mathbf{F})_{mm}^{-1}\mathbf{M}(\mathbf{F})_{mn}\mathbf{k}_{t+1} \text{ and}$$

$$\mathbf{q}_0 = -\mathbf{N}(\mathbf{F})_{mn}\mathbf{k}_0 = -\mathbf{M}(\mathbf{F})_{mm}^{-1}\mathbf{M}(\mathbf{F})_{mn}\mathbf{k}_0.$$

Identification restrictions so that the rule depends only upon pre-determined variables: restricted «quasi-optimal» rules

$$r_t - r^* = -\mathbf{F}_{1n}\left(\frac{\mathbf{k}_{t+1}-\mathbf{k}^*}{\mathbf{k}^*}\right) - \mathbf{F}_{1m}\left(\frac{\mathbf{q}_{t+1}-\mathbf{q}^*}{\mathbf{q}^*}\right)$$
$$= -\underbrace{\left(\mathbf{F}_{1n}-\mathbf{F}_{1m}\mathbf{N}_{mn}\right)}_{\mathbf{F}'_{1n}}\left(\frac{\mathbf{k}_{t+1}-\mathbf{k}^*}{\mathbf{k}^*}\right),$$
$$\mathbf{F} = \left(\mathbf{F}_{1n}, \mathbf{F}_{1m}\right) = \left(\mathbf{F}_{1n}-\mathbf{F}_{1m}\mathbf{N}_{mn}, \mathbf{0}_{1m}\right).$$

Optimal control program dimension N<N+M

The policy-maker only needs to control predetermined variables:

$$\max_{\{R_t\}} -\frac{1}{2}\sum_{t=0}^{+\infty}\beta'^t\left[\mathbf{Q}'_{nn}\left(\frac{\mathbf{k}_t-\mathbf{k}^*}{\mathbf{k}^*}\right)^2+\rho\left(r_t-r^*\right)^2\right]$$

subject to the closed loop system of pre-determined variables:

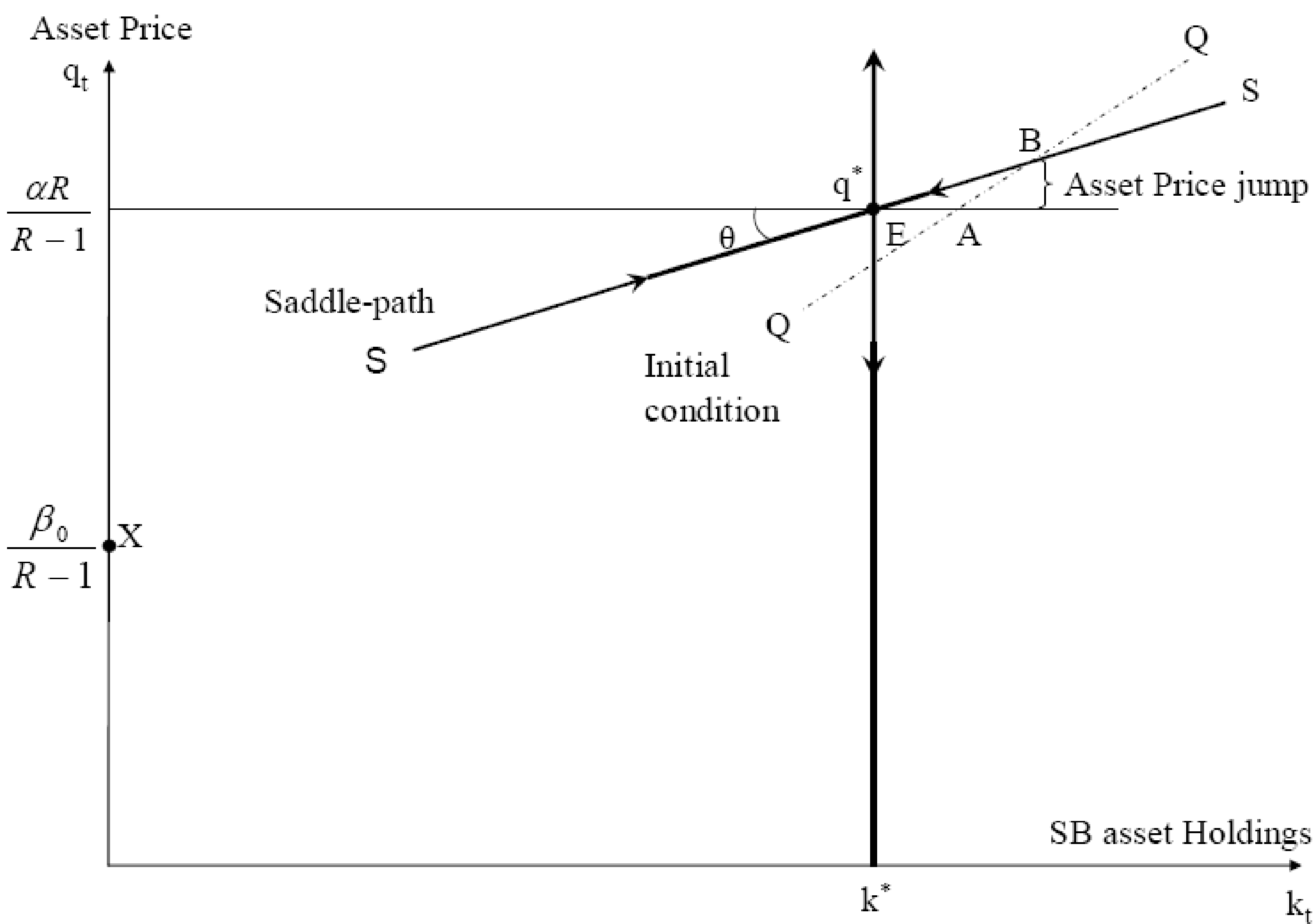
$$\mathbf{k}_{t+1}=\left(\mathbf{A}'_{nn}-\mathbf{B}_{n1}\mathbf{F}'_{1n}\right)\mathbf{k}_t.$$

According to the following equality:

$$\mathbf{A}'_{nn}=\mathbf{A}_{nn}-\mathbf{A}_{nm}\mathbf{N}_{mn} \text{ and } \mathbf{F}'_{1n}=\mathbf{F}_{1n}-\mathbf{F}_{1m}\mathbf{N}_{mn}$$

$$\mathbf{Q}'_{nn}=\mathbf{Q}_{nn}+\mathbf{N}\left(\mathbf{F}\right)^T_{nm}\mathbf{Q}_{mm}\mathbf{N}\left(\mathbf{F}\right)_{mn}+\mathbf{Q}_{nm}\mathbf{N}\left(\mathbf{F}\right)_{mn}+\mathbf{N}\left(\mathbf{F}\right)^T_{nm}\mathbf{Q}_{mn}$$

Evaluate the augmented Taylor rule on the stable manifold (here, a line)



Instantaneous jump immediately after a shock on productivity A on the stable manifold (here a line)

$$q_0 - q^* \neq \frac{a}{R - \frac{1}{2}} \left(k_0 - k^* \right)$$

$$\prime \qquad\qquad\qquad\qquad \prime$$

$$q_0' - q^* = \frac{a}{R - \frac{1}{2}} \left(k_0 - k^* \right).$$

Identification

If 2 different values of a parameter (for example **zero, positive or negative**)

Leads to an observationally equivalent model to be tested with data:

Non identification

Textbook demand supply example: Two endogenous variables quantity and price (with 4 parameters to estimate) in a system of two equations and only one exogenous variable (a constant). Reduced form cannot estimate all 4 parameters.

For ad hoc rules, no optimisation on pre-determined variables. M ad hoc identification restrictions on rules parameters may set N rule parameters for k(t) equal to zero, ok if M ≤ N, problem if N<M

$$r_t - r^* = -\mathbf{F}_{1n}\left(\frac{\mathbf{k}_{t+1}-\mathbf{k}^*}{\mathbf{k}^*}\right) - \mathbf{F}_{1m}\left(\frac{\mathbf{q}_{t+1}-\mathbf{q}^*}{\mathbf{q}^*}\right)$$

$$= -\underbrace{(\mathbf{F}_{1n} - \mathbf{F}_{1m}\mathbf{N}_{mn})}_{\mathbf{F}'_{1n}}\left(\frac{\mathbf{k}_{t+1}-\mathbf{k}^*}{\mathbf{k}^*}\right),$$

$$\mathbf{F} = (\mathbf{F}_{1n}, \mathbf{F}_{1m}) = (\mathbf{F}_{1n} - \mathbf{F}_{1m}\mathbf{N}_{mn}, \mathbf{0}_{1m}).$$

Identification in estimated DSGE

Ex-ante and Ex-post evaluations of identification (Kemunjer and Ng (2011), Econometrica, Cochrane (2011) on Taylor rule, J Pol Eco).

Auto-regressive shocks parameter: identified.

Taylor rule parameters and many other DSGE parameters: not identified.

Indeterminacy of Blanchard Kahn « unique » solution: Blake Kirsanova (2012)

To select quasi-optimal rules with n stable dimensions

You have the choice between n+m stable dimensions

There are the number to choose a set of n elements in a set of N+M « stable eigenvectors ».

for building matrix –N=C

Conclusion for quasi-optimal rules

1. Lack of identification of rule parameters for non predetermined variables
2. Indeterminacy
3. Time consistency à la Calvo.
4. Covariance matrix between non predetermined and predetermined variables is fixed (but non unique), without ANY effect of policy.

4. Over stable Optimal Rules

Compromise: Rational expectations

Over-stable (as Old Keynesian and Kalman)

Determinacy

Precommitment (time inconsistency problem)

Pretermined Lagrange multipliers of non predetermined variables

$$-\frac{1}{2}\sum_{t=0}^{+\infty}\beta^t\left[\begin{array}{c}\left(\frac{\mathbf{k}_t-\mathbf{k}^*}{\mathbf{k}^*}\right)^T\mathbf{Q}_{nn}\left(\frac{\mathbf{k}_t-\mathbf{k}^*}{\mathbf{k}^*}\right)+\left(\frac{\mathbf{q}_t-\mathbf{q}^*}{\mathbf{q}^*}\right)^T\mathbf{Q}_{mm}\left(\frac{\mathbf{q}_t-\mathbf{q}^*}{\mathbf{q}^*}\right)\\+\left(\frac{\mathbf{k}_t-\mathbf{k}^*}{\mathbf{k}^*}\right)^T\mathbf{Q}_{nm}\left(\frac{\mathbf{q}_t-\mathbf{q}^*}{\mathbf{q}^*}\right)+\left(\frac{\mathbf{q}_t-\mathbf{q}^*}{\mathbf{q}^*}\right)^T\mathbf{Q}_{mn}\left(\frac{\mathbf{k}_t-\mathbf{k}^*}{\mathbf{k}^*}\right)\\+\rho\left(r_t-r^*\right)^2+2\beta\mu_{t+1}^T\left(\mathbf{A}\mathbf{y}_t+\mathbf{B}\left(r_t-r^*\right)-\mathbf{y}_{t+1}\right)\end{array}\right] \tag{19}$$

where $2\beta'\mu_{t+1}$ is the Lagrange multiplier associated with the linear constraint. First order conditions with respect to r_t and $\mathbf{y}_t$, respectively, are:

$$0=\rho\left(r_t-r^*\right)+\beta\mathbf{B}^T\mu_{t+1} \tag{20}$$

$$\mu_t=\mathbf{Q}\mathbf{y}_t+\beta\mathbf{A}^T\mu_{t+1} \tag{21}$$

« As if » non pre-determined are pre-determined

Ljungqvist and Sargent (2012, chapter 19) describe a four step algorithm for solving the optimal policy under commitment. "*Step 1 seems to disregard the forward looking aspect of the problem. If we temporarily ignore the fact that the* $\mathbf{q}_0$ *component of the state* $\mathbf{y}_0 = \begin{pmatrix} \frac{\mathbf{k}_0 - \mathbf{k}^*}{\mathbf{k}^*} \\ \frac{\mathbf{q}_0 - \mathbf{q}^*}{\mathbf{q}^*} \end{pmatrix}$ *is not actually a state vector, then superficially the Stackelberg problem has the form of an optimal linear regulator problem*" (Ljungqvist and Sargent (2012, chapter 19, p.769).

The rule depends on Lagrange multiplier of non predetermined variables (besides the « as if » explicit rule)

Step 3 uses the property that a stabilizing solution satisfies:

$$\mu_t = \begin{pmatrix} \mu_{\mathbf{k},t} \\ \mu_{\mathbf{q},t} \end{pmatrix} = \mathbf{P} \begin{pmatrix} \mathbf{k}_t \\ \mathbf{q}_t \end{pmatrix} = \begin{pmatrix} \mathbf{P}_{nn} & \mathbf{P}_{nm} \\ \mathbf{P}_{mn} & \mathbf{P}_{mm} \end{pmatrix} \begin{pmatrix} \mathbf{k}_t \\ \mathbf{q}_t \end{pmatrix}, \forall t \in \mathbb{N}$$

$$\mathbf{q}_t = \begin{pmatrix} -\mathbf{P}_{mm}^{-1}\mathbf{P}_{mn} & \mathbf{P}_{mm}^{-1} \end{pmatrix} \begin{pmatrix} \mathbf{k}_t \\ \mu_{\mathbf{q},t} \end{pmatrix} \text{ and } \mathbf{q}_0 = -\mathbf{P}_{mm}^{-1}\mathbf{P}_{mn}\mathbf{k}_0 \text{ if } \mu_{\mathbf{q},t=0} = 0$$

$$r_t = \mathbf{\Phi} \begin{pmatrix} \mathbf{k}_t \\ \mu_{\mathbf{q},t} \end{pmatrix} = -\mathbf{F} \begin{pmatrix} \mathbf{I}_{nn} & \mathbf{0}_{nm} \\ -\mathbf{P}_{mm}^{-1}\mathbf{P}_{mn} & \mathbf{P}_{mm}^{-1} \end{pmatrix} \begin{pmatrix} \mathbf{k}_t \\ \mu_{\mathbf{q},t} \end{pmatrix}$$

$$\begin{pmatrix} \mathbf{k}_{t+1} \\ \mu_{\mathbf{q},t+1} \end{pmatrix} = \begin{pmatrix} \mathbf{I}_{nn} & \mathbf{0}_{nm} \\ \mathbf{P}_{mn} & \mathbf{P}_{mm} \end{pmatrix} (\mathbf{A} - \mathbf{BF}) \begin{pmatrix} \mathbf{I}_{nn} & \mathbf{0}_{nm} \\ -\mathbf{P}_{mm}^{-1}\mathbf{P}_{mn} & \mathbf{P}_{mm}^{-1} \end{pmatrix} \begin{pmatrix} \mathbf{k}_t \\ \mu_{\mathbf{q},t} \end{pmatrix}$$

If Kalman controllability in N+M dimensions with distinct eigenvalues

Each non pre-determined variables has its own specific freedom to vary distinctly from other variables (N+M dimensional system).

For the LQR, there is a unique correspondance between the set of eigenvalues and the set of parameters of the rule.

This allows identification of the rule parameters.

Determinacy with P from optimal decision

4. Determinacy. Kalman's controllability condition is a precondition for assuming that the Lagrange multipliers related to non predetermined variables should be all equal to zero at the initial date $\mu_{\mathbf{q},t=0} = 0$ *(Bryson and Ho (1975), p.55-59; Xie (1997) provides a counter example where Kalman's controllability condition is not satisfied). As the Lagrange multipliers are related to the optimal value function matrix as follows:* $\mu_{\mathbf{z},t} = \mathbf{P}\mathbf{z}_t$*,the initial values of non-predetermined variables are linear functions of the initial values of predetermined variables (Ljundqvist and Sargent's (2012, Chapter 19), Jensen (2011)):*

$$\mathbf{q}_0 = -\mathbf{P}_{mm}^{-1}\mathbf{P}_{mn}\mathbf{k}_0 \text{ if } \mu_{\mathbf{q},t=0} = 0. \tag{25}$$

Additional M degrees of freedom to explain phenomena

6. Minimal volatility of the policy interest rate ($\rho > 0, \mathbf{Q} = \mathbf{0}$)*. It is such that stable eigenvalues of the open loop system are the same as in the closed loop system* $|\lambda_{i,\mathbf{A-BF}}| = |\lambda_{i,\mathbf{A}}| < 1$ *and that unstable eigenvalues (indexed by* $i\prime$*) of the open loop system are mirrored by stable eigenvalues in the closed loop system having their modulus such that* $|\lambda_{i',\mathbf{A-BF}}| = 1/|\lambda_{i',\mathbf{A}}| < 1$ *(Rojas (2011)).*

7. Ability of policies to decrease the covariances matrix between pre-determined and non pre-determined variables when the policy maker preferences are such that $\mathbf{Q}_{mn} \neq \mathbf{0}$ *and* $\mathbf{Q}_{nm} \neq \mathbf{0}$*.*

Time consistency problem à la Calvo (1978)

8. Time inconsistency à la Calvo (1978). When the system is controllable and without a pre-commitment constraint, a policy maker who optimize again on period $t+1$ *would choose an initial condition* $\mu_{\mathbf{q},t+1} = \mathbf{0}$ *instead of the optimal path* $\mu_{\mathbf{q},t+1} \neq \mathbf{0}$ *decided on date* t*. The system remains bounded and stable if ever the policy maker chooses* $\mu_{\mathbf{q},t+k} = \mathbf{0}$ *on all following periods*

p(t) is the Lagrange multiplier of a non pre-determined variable, p(t=0)=0

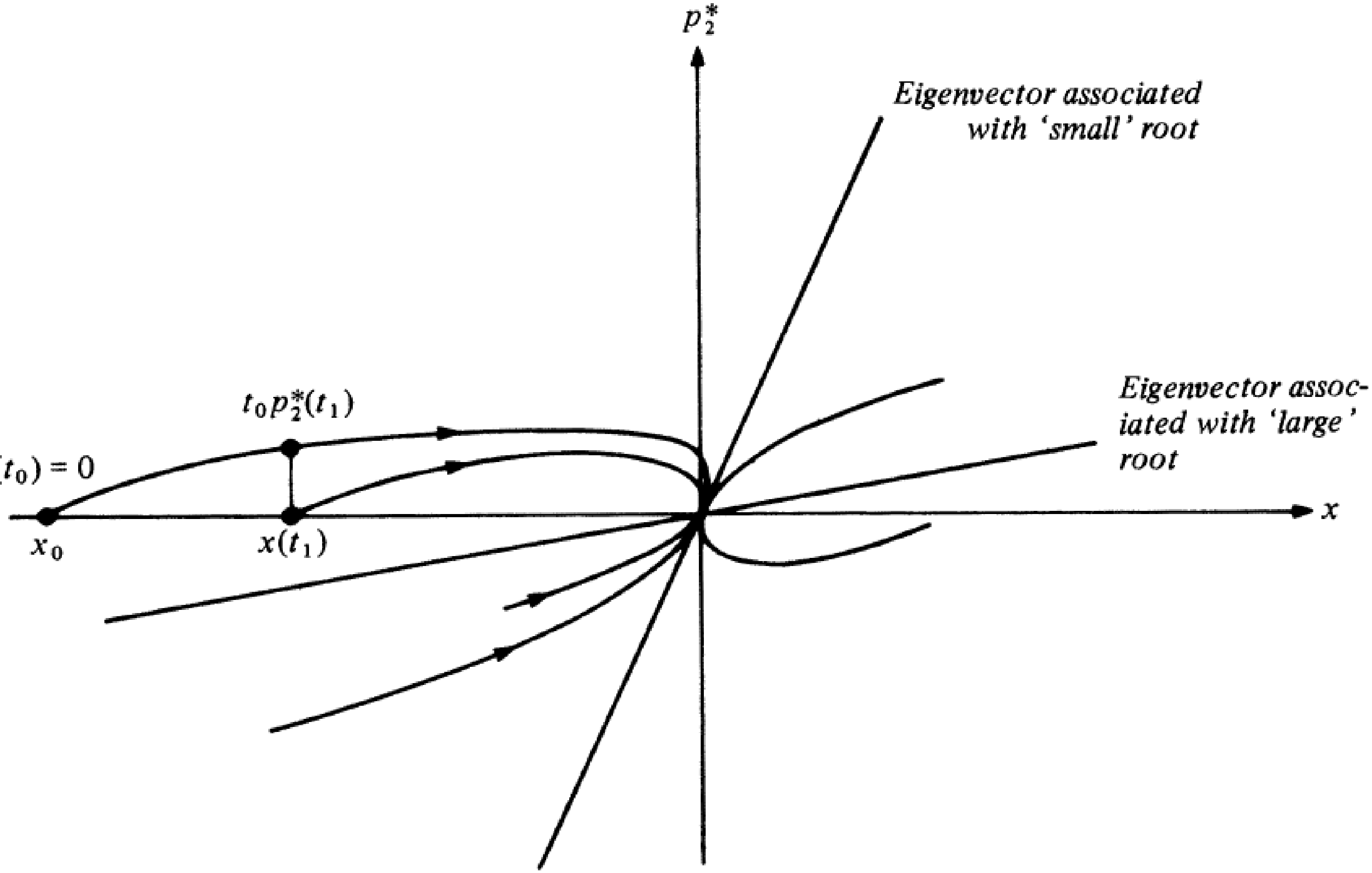


Fig. 1: The 'time inconsistency' of optimal policy

OK, but what do I need to change in my models?... A few signs of **$dX(t+1)/dr(t)$**.

Bubbles > sunspots DSGE models are designed so that the rule is not stabilizing in some dimensions, so that for example:

$b = dX(t+1)/dr(t) > 0$,

$f = dr(t)/dX(t) > 0$

So that **$a < 1 < a + bf$**

It happens in some cases that you need to turn unstable after control a dimension which is stable before control in order to maintain exactly N stable dimensions and M unstable dimensions.

$X(t+1)$ output gap, asset prices, financial market vulnerability.

An example of over stable rule with unexpected sign

$X(t) - X(t+1) = -\mathbf{b}\, r(t)$

Output gap X(t) at date t is a <u>negative</u> function of the interest rate.

Output gap X(t+1) at date t+1 is a <u>positive</u> function of the interest rate.

Then, a stabilizing optimal rule is a negative function of the output gap:

$r(t) = \mathbf{-\ 0.05}$ output gap(t) + 1.6 . Inflation (t)

(1) Stabilizing **M** unstable dimensions

$X(t+1) = a \, . \, X(t) + \mathbf{b} r(t), \quad r(t) = \mathbf{f} . X(t)$

$X(t+1) = (a + \mathbf{bf}) . X(t)$

IF $b \neq 0$ (Kalman controllability; scalar case);

$b = dX(t+1)/dr(t) < 0$, $f = dr(t)/dX(t) > 0$ OR

$b = dX(t+1)/dr(t) > 0$, $f = dr(t)/dX(t) < 0$

$$0 < a + \mathbf{bf} < 1 < a$$

5. Optimal rules
robust to mis-specification

Blanchard Kahn lack of optimal control robustness when the expectations of errors in (1b) is not zero, even with an infinitesimal deviation from zero

« Robust » optimal control takes into account the risk of « misspecified » model:

- omitted variable bias
- biased measurement errors on inflation, on the output gap.
- endogeneity bias

THEN: infinitesimal deviation in (1b): the economy blows up with hyperinflation, deflation, depression overheating, bubbles and krachs for credit and asset prices.

M non pre-determined are « determined » by N pre-determined, immediate self-correction to shocks in M dimension to remain on the stable manifold. The expectations of errors in (1b) is ALWAYS ZERO.

$$\text{(1a)} \qquad \underbrace{X_{t+1}}_{n_X \times 1} = \underbrace{A(\theta)}_{n_X \times n_X} \underbrace{X_t}_{n_X \times 1} + \underbrace{B(\theta)}_{n_X \times n_\varepsilon} \underbrace{\varepsilon_{t+1}}_{n_\varepsilon \times 1},$$

$$\text{(1b)} \qquad \underbrace{Y_{t+1}}_{n_Y \times 1} = \underbrace{C(\theta)}_{n_Y \times n_X} \underbrace{X_t}_{n_X \times 1} + \underbrace{D(\theta)}_{n_Y \times n_\varepsilon} \underbrace{\varepsilon_{t+1}}_{n_\varepsilon \times 1}.$$

Macro-prudential DSGE is a non cumulative literature with financial stability results which are not robust to mis-specification

A second macro-prudential DSGE model

which adds an omitted financial market vulneratibility variable **$\eta(t)$**

with respect to a first macro-prudential DSGE

$q(t)=C.k(t)+\varepsilon+\mathbf{D.\eta(t)}$

Demonstrates that the first macro-prudential model model was blowing up the economy when omitting **$D.\eta(t)$, non-zero mean of disturbances**

Linear system multiple equilibria, crisis and **robust** macroprudential policy

If the system has a saddlepoint finite long term equilibrium **x***,

It has also two alternative multiple equilibria for *m* diverging dimensions with two alternative long term equilibria: **+∞ and -∞**. (or **zero** if the variable is bounded below And **xmax** if the variable is bounded upwards).

Robust preventive macro-prudential policy should avoid the crisis with **policy** leaning against those potential bubbles in the neighbourhood of **x(0)** avoiding the bad extreme equilibria.

Robust preventive macro-prudential policy

Does not deal with the exit of a bad equilibrium to get out of a financial crisis.

Holds only in the neigthbourhood of a « good » long run equilibrium with a linearized system valid for relative deviations of variables from their long run equilibrium value at most equal to 10%.

Robust macro-prudential policy

Min-max of losses **+∞ and -∞** when the expectations of disturbances is distinct from zero in equation (1b).

This implies seeking for bounded solutions, « over-stable » rules in the min-max optimization

And min-max the finite losses when the expectations of disturbances is distinct from zero in equation (1a).

Example: Probability of Stable Control of an Unstable Plant

- Longitudinal dynamics for a Forward-Swept-Wing Demonstrator

$$F = \begin{bmatrix} -2gf_{11}/V & \rho V^2 f_{12}/2 & \rho V f_{13} & -g \\ -45/V^2 & \rho V f_{22}/2 & 1 & 0 \\ 0 & \rho V^2 f_{32}/2 & \rho V f_{33} & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}; \quad G = \begin{bmatrix} g_{11} & g_{12} \\ 0 & 0 \\ g_{31} & g_{32} \\ 0 & 0 \end{bmatrix}; \quad x = \begin{bmatrix} V \\ \alpha \\ q \\ \theta \end{bmatrix}$$

- Nominal eigenvalues (one unstable)

$$\lambda_{1-4} = -0.1 \pm 0.057j, \quad -5.15, \quad 3.35$$

Air density and airspeed, ρ and V, have uniform distributions($\pm 30\%$)

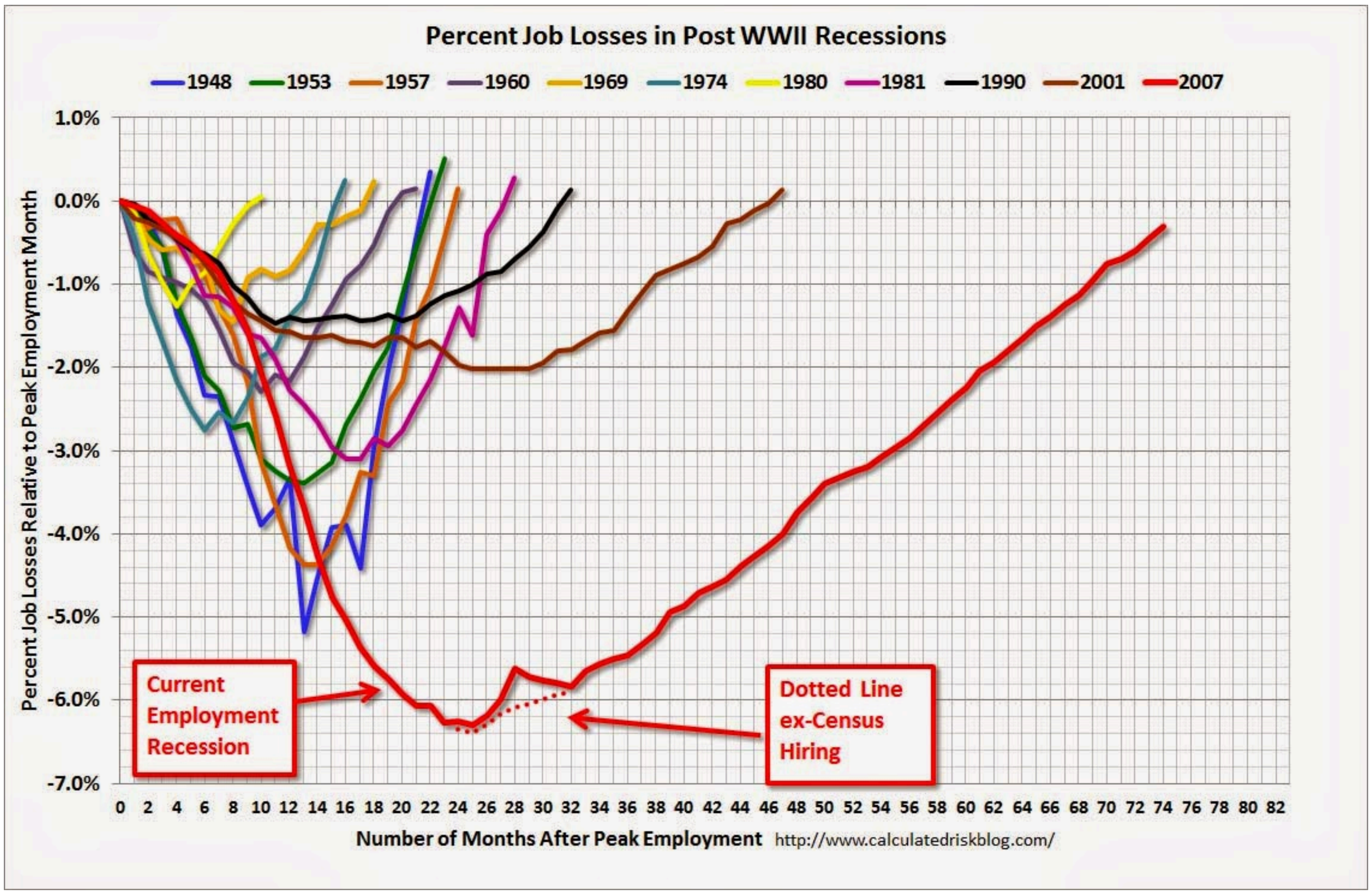
10 coefficients have Gaussian distributions ($\sigma = 30\%$)

$$\mathbf{p} = \begin{bmatrix} \rho & V & f_{11} & f_{12} & f_{13} & f_{22} & f_{32} & f_{33} & g_{11} & g_{12} & g_{31} & g_{32} \end{bmatrix}^T$$

Environment | Uncontrolled Dynamics | Control Effect

Applying Blanchard and Kahn (1980) unique solution, maintaining instability in *m* dimensions, with **instantaneous, not modelled, self adjustment** of the system to shocks in those *m* dimensions,
with lack of robustness to infinitesimal deviation of the mean of errors of specification in those *m* dimenions

Robust to misspecification, preventive, macro-prudential policy goal is to avoid crisis (USA, April 2004)



LQ Regulators for the Example

- Three stabilizing feedback control laws

- Case a) LQR with low control weighting

$$Q = diag(1,1,1,0); \quad R = (1,1); \quad \lambda_{\text{[illegible]}} = -35, -6.1, -3.3, -.02$$

$$C = \begin{bmatrix} 0.17 & 130 & 33 & 0.36 \\ 0.98 & -11 & -3 & -1.1 \end{bmatrix}$$

- Case b) LQR with high control weighting

$$Q = diag(1,1,1,0); \quad R = (1000,1000); \quad \lambda_{\text{[illegible]}} = -5.2, -3.4, -1.1, -.02$$

$$C = \begin{bmatrix} 0.03 & 83 & 21 & -0.06 \\ 0.01 & -63 & -16 & -1.9 \end{bmatrix}$$

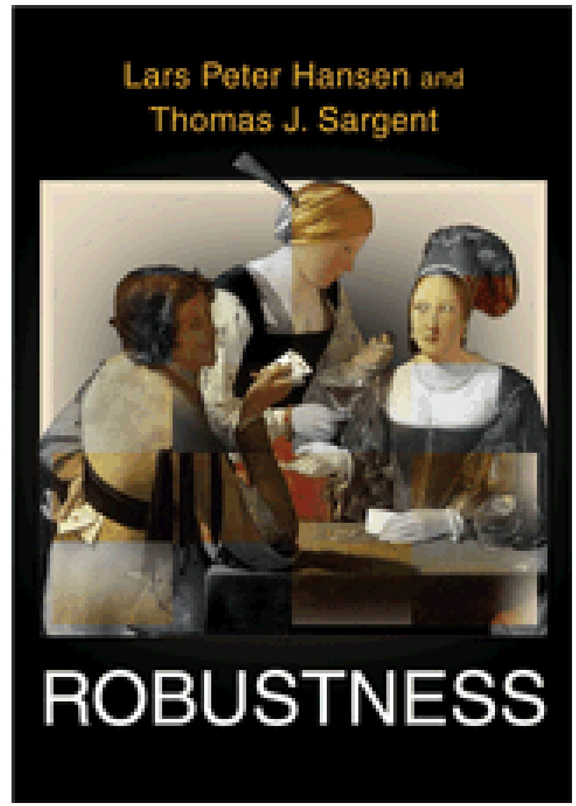
- Case c) Case b with gains multiplied by 5 for bandwidth (loop-transfer) recovery

$$\lambda_{\text{[illegible]}} = -32, -5.2, -3.4, -0.01$$

$$C = \begin{bmatrix} 0.13 & 413 & 105 & -0.32 \\ 0.05 & -313 & -81 & -1.1 - 9.5 \end{bmatrix}$$

Robust optimal control by Hansen Sargent (2007)

Robust rules obtained in minimizing the maximum of losses (with parameter θ measuring mis-specification aversion) when the expectations of errors is non zero for (1a) and (1b).



Conclusion
Macro-prudential DSGE models: From Blanchard Kahn to Kalman's controllability?

Kalman's US national medal of science (2009) and his contribution to Apollo program