Consistent Aggregation
With Superlative and Other Price Indices

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Abstract

Various fields of economic analysis (e.g., growth and productivity) and economic policy (e.g., monetary and social policy) rely on accurate measures of price change. Unfortunately, the price index formulae that most price statisticians consider as particularly accurate – the superlative indices of Fisher, Törnqvist, and Walsh – are believed to violate the property of consistency in aggregation. This property, however, is indispensable for economic studies that attempt to disaggregate the overall result into the contributions of individual entities such as sectors of the economy or groups of products. The present paper introduces a thoroughly motivated formal definition of consistency in aggregation and proves that, contrary to general perception, the three superlative price indices can be considered as consistent in aggregation. Furthermore, many other price indices are shown to be consistent in aggregation. The theoretical findings are applied to the Swedish consumer price index.

Keywords: Consistent aggregation, index theory, superlative price index, decomposition, CPI

JEL classification: C43, E31

1. Introduction

In most fields of applied economic analysis, the diversity of individual changes must be aggregated into some single number measuring the average change. Prominent examples are the changes in national income, unemployment, money supply, or prices. To provide additional insights, the computation of the average change is often

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conducted in a two stage procedure, where on the first stage average changes of subgroups are computed and on the second stage these individual results are aggregated into the overall change.

For example, some central banks and many financial analysts decompose the universe of consumer products into the "core products" (all products except for energy and seasonal food) and the "non-core products" (energy and seasonal food). The average price change of the core products is called the core inflation, whereas the average price change of the non-core products could be denoted as the non-core inflation.

Core inflation is often considered as a measure of the long-run inflation trend, and therefore, as a key indicator for monetary policy. Averaging the core inflation and the non-core inflation yields the overall inflation rate of the economy. The overall inflation rate is often used for indexing various types of contracts and for transforming nominal values into real values (e.g., national income). The separate compilations of the core inflation and the non-core inflation reveal how strongly the economy's current overall inflation is driven by its long-run inflation trend.[1]

Alternatively, the overall inflation rate could be directly computed from the complete universe of products, without decomposing this universe into the core products and the non-core products. This single stage computation is simpler, but provides fewer insights.

The calculation of an average price change is accomplished by some price index formula. It is considered as a major advantage of a price index formula, when it computes the same overall inflation, regardless of whether it is applied in a single stage or two stage calculation. When a price index formula satisfies this postulate, the formula is denoted as *consistent in aggregation*.

The notion of consistency in aggregation has been alluded to by van Yzeren (1958). Vartia (1976a, b) fleshed out this notion and Blackorby and Primont (1980) formalized and generalized it. Stuvel (1989) and Balk (1995, 1996, 2008) take the position that in the field of price measurement the general notion proposed by Blackorby and Primont is not adequate. They suggests to revert to Vartia's narrow notion. Auer (2004) contests this position and advocates a definition of consistency of aggregation that is more general than the definition of Balk and Vartia, but less general than that of Blackorby and Primont. Pursiainen (2005, 2008) provides a rigorous formal treatment of Vartia's original notion.

All of these studies agree that consistency in aggregation of a price index not only

[1]There is a vivid controversy on the rationale for using the core inflation as a yardstick for monetary policy (e.g., Crone *et al.*, 2013). The present paper, however, is not concerned with this controversy, but merely uses the notions of core-inflation and non-core inflation to illustrate the process and the benefits of a two stage computation.

requires that the single stage and the two stage computations yield the same outcome, but also that three additional conditions (details are spelled out below) must be satisfied. In the following, this common view is referred to as the "four consensus conditions" of consistency in aggregation. As the above dispute suggests, however, endorsement of the consensus conditions still leaves much space for disagreement.

The present paper's first main contribution is an attempt to settle this dispute. For this purpose it develops a thoroughly motivated new definition of consistency in aggregation. It is more general than those proposed by Auer (2004, p. 390), Balk (1995, p. 85; 1996, pp. 358-60), Pursiainen (2005, p. 21; 2008, p. 18), Stuvel (1989, p. 36), van Yzeren (1958, p. 432), and Vartia (1976a, pp. 85-89; 1976b, p. 124). The analysis reveals that several attractive price indices that, hitherto, have been perceived as violating the consensus conditions, turn out to be fully consistent with these conditions. In other words, these price indices are consistent in aggregation, and therefore, perfectly appropriate for multi stage computations in applied empirical analysis.

It would be useful, if among these price indices were a *superlative* one. The concept of superlative price indices has been introduced by Diewert (1976). These indices are often advocated as generating particularly reliable results and being firmly anchored in economic theory. The most vehemently recommended superlative price indices are the Fisher, Törnqvist, and Walsh indices. However, these indices have been perceived as not being consistent in aggregation (e.g., Diewert, 2004a, pp. 349-350; Balk, 2008, pp. 110-111). The present paper proves that this perception cannot be maintained in the context of our new definition of consistency in aggregation. This is the second contribution of this study. As a third contribution it shows that many other price indices are consistent in aggregation.

This paper is organized as follows. Section 2 provides a more detailed account of the consensus conditions. Section 3 leaves the field of price indices and develops a precise mathematical definition of consistent aggregation rules. In Section 4 we return to the analysis of price indices and provide a rigorous mathematical definition of price indices. In Section 5 we apply this definition to a multi stage computation of the Swedish consumer price index. The results suggest that superlative price indices may well be consistent in aggregation. Section 6 connects the theoretical concepts developed in Sections 3 and 4. It presents a rigorous mathematical definition of a price index that is consistent in aggregation. In Section 7 we examine whether superlative price indices exist that are consistent in aggregation. In Section 8 we show that many (non-superlative) price indices are consistent in aggregation. An application to the Swedish price data is presented in Section 9. For practical price measurement purposes additional requirements can be attached to our definition of consistency in aggregation. Section 10 explains these requirements, examines which price index formulae satisfy these requirements, and relates the new definition of consistency in aggregation to alternative definitions proposed in the literature. Concluding remarks are contained in Section 11.

2. Two Stage Computation of Price Changes

In an economy a vast number of goods and services are sold. The prices of these "items" change over time. A price index attempts to measure the items' average price change between a base and a comparison period. It is assumed that during both periods all N items are available and that their prices and quantities are correctly recorded. Let D denote the finite set containing the N items' prices and quantities. A price index formula P (e.g., Laspeyres index) is usually considered as a function that maps the recorded prices and quantities into a single positive number that indicates the N items' average price change.

The *single stage computation* of the overall price change applies a given price index formula P to the complete set D. In contrast, a *two stage computation* starts by partitioning the set D into several subsets D_k. For each of these subsets a price index P_k is computed. In the second stage of this two stage procedure, the numbers P_k are aggregated into the overall price change. Such a two stage computation provides important additional insights, because it allows to identify the individual forces driving the overall result. Of course, the single stage and the two stage computations should be "consistent". The precise meaning of "consistency", however, is difficult to define.

As pointed out before, some consensus conditions exist that spell out the meaning of consistency in more detail. These consensus conditions relate only to price indices P that are continuous in the prices and quantities (continuity axiom) and that are invariant with respect to changes in the units in which the quantities are measured (commensurability axiom).[2] For such price indices the consensus conditions specify how a two stage computation should be conducted and how it should relate to the single stage computation (e.g., Auer, 2004, p. 385; Balk, 1995, p. 85; Balk, 1996, pp. 358-59; Balk, 2008, pp. 108-109; Vartia, 1976b, p. 124):

(i) For all possible partitions of the set D, the two stage computation of the overall price change of D must yield the same index number as the single stage computation.

(ii) On both stages of the two stage computation the "same index formula" must be

[2]There is a large body of literature discussing the axioms a sensible price index formula should satisfy. In his comprehensive survey Diewert (2004b, pp. 293-294) points out that the continuity axiom is informally discussed in Fisher (1922, pp. 207-215) and that the commensurability axiom can be traced back to Jevons (1865, p. 23) and Pierson (1896, p. 131).

applied as in the single stage computation (only the number of variables can be different).

(iii) In the first stage of the two stage computation, for each subset D_k, a price index number P_k and one or more aggregate values are computed. The price index numbers P_k and the aggregate value(s) depend only on the prices and quantities of the items in subset D_k.

(iv) In the second stage of the two stage computation, the index number for the complete set D depends only on the index numbers P_k and the aggregate values computed in the first stage computations.

Unfortunately, these consensus conditions leave much space for ambiguity and disagreement. It is therefore necessary to transform the four consensus conditions into a thoroughly motivated formal definition of consistency in aggregation. As a preliminary step we develop the notion of a consistent aggregation rule.

3. Consistent Aggregation Rules

An aggregation rule is a procedure which aggregates a finite set of data into a single datum. The paradigmatic example is the sum of finitely many numbers (or vectors) where, for each size of the data set, the procedure has the "same functional form". Formally, the expression "same functional form" does not make much sense as, e.g., the maps $A_2 : \mathbb{R}^2 \to \mathbb{R}$, $(d_1, d_2) \mapsto d_1 + d_2$ and $A_3 : \mathbb{R}^3 \to \mathbb{R}$, $(d_1, d_2, d_3) \mapsto d_1 + d_2 + d_3$ have different domains, and therefore, are totally different objects. The link between A_2 and A_3 is the law of associativity

$$A_3(d_1, d_2, d_3) = A_2(A_2(d_1, d_2), d_3) = A_2(d_1, A_2(d_2, d_3)) ,$$

for all $d_1, d_2, d_3 \in \mathbb{R}$. As we will see, consistency of an aggregation rule is just a slight generalization.

Let I be any set of possible data $\mathbf{d}$ (typically belonging to some $\mathbb{R}^k$).

Definition 1. An aggregation rule for the set I is a sequence $A = (A_n)_{n\in\mathbb{N}}$ of maps

$$A_n : I^n \to I , \quad (\mathbf{d}_1, \dots, \mathbf{d}_n) \mapsto A_n(\mathbf{d}_1, \dots, \mathbf{d}_n)$$

with $A_1(\mathbf{d}_i) = \mathbf{d}_i$ for all $\mathbf{d}_i \in I$.

However, Definition 1 covers also meaningless aggregation rules such as $A_n(\mathbf{d}_1, \dots, \mathbf{d}_n) = \mathbf{d}_1$. To come closer to the intuitive idea of a meaningful aggregation rule, one needs further properties. In the first place, a meaningful aggregation rule requires that the ordering of the data vector $(\mathbf{d}_1, \ldots, \mathbf{d}_n) \in I^n$ is irrelevant, that is, A_n are symmetric. Secondly, one would expect that the aggregation of $(\mathbf{d}_1, \ldots, \mathbf{d}_n) \in I^n$ in one step, $A_n(\mathbf{d}_1, \ldots, \mathbf{d}_n)$, yields the same result as the following procedure: (1) $(\mathbf{d}_1, \ldots, \mathbf{d}_n)$ is partitioned into two arbitrary "groups" $(\mathbf{d}_1, \ldots, \mathbf{d}_m) \in I^m$ and $(\mathbf{d}_{m+1}, \ldots, \mathbf{d}_{m+k}) \in I^k$, with $n = m + k$; (2) $A_m(\mathbf{d}_1, \ldots, \mathbf{d}_m)$ and $A_k(\mathbf{d}_{m+1}, \ldots, \mathbf{d}_{m+k})$ are computed; (3) these results are treated as new data and aggregated by A_2.

These two requirements for a meaningful aggregation rule can be summarized in the following definition:

Definition 2. An aggregation rule $A = (A_n)_{n\in\mathbb{N}}$ is *consistent,* if it is symmetric and

$$A_n(\mathbf{d}_1, \ldots, \mathbf{d}_n) = A_2(A_m(\mathbf{d}_1, \ldots, \mathbf{d}_m), A_k(\mathbf{d}_{m+1}, \ldots, \mathbf{d}_{m+k})) \ , \tag{1}$$

for all $m, k \in \mathbb{N}$ with $n = m + k$, $(\mathbf{d}_1, \ldots, \mathbf{d}_m) \in I^m$, and $(\mathbf{d}_{m+1}, \ldots, \mathbf{d}_n) \in I^k$.

As another example of an aggregation rule for $I = \mathbb{R}$ consider the calculation of a product:

$$A_n(d_1, \ldots, d_n) = \prod_{i=1}^{n} d_i \ , \tag{2}$$

with $d_1, \ldots, d_n \in \mathbb{R}$. This aggregation rule is symmetric. Since for all $m, n \in \mathbb{N}$,

$$\prod_{i=1}^{n} d_i = (d_1 \cdot \ldots \cdot d_m) \cdot (d_{m+1} \cdot \ldots \cdot d_n) \ ,$$

the aggregation rule also satisfies condition (1). Therefore, it is a consistent aggregation rule.

Recall that a binary operation $\oplus$ on I, i.e. a function $I^2 \to I$, $(\mathbf{d}_1, \mathbf{d}_2) \mapsto \mathbf{d}_1 \oplus \mathbf{d}_2$, is commutative and associative, if $\mathbf{d}_1 \oplus \mathbf{d}_2 = \mathbf{d}_2 \oplus \mathbf{d}_1$ and $(\mathbf{d}_1 \oplus \mathbf{d}_2) \oplus \mathbf{d}_3 = \mathbf{d}_1 \oplus (\mathbf{d}_2 \oplus \mathbf{d}_3)$ for all $\mathbf{d}_1$, $\mathbf{d}_2$, $\mathbf{d}_3 \in I$.

Proposition 1. *$A = (A_n)_{n\in\mathbb{N}}$ is a consistent aggregation rule for I, if and only if some commutative and associative binary operation $\oplus_A$ exists, such that*

$$A_n(\mathbf{d}_1, \ldots, \mathbf{d}_n) = \mathbf{d}_1 \oplus_A \mathbf{d}_2 \oplus_A \cdots \oplus_A \mathbf{d}_n \ , \tag{3}$$

for all $n \in \mathbb{N}$ and $\mathbf{d}_1, \ldots, \mathbf{d}_n \in I$. For $n = 1$, the right hand side of (3) is interpreted as $\mathbf{d}_1$.[3]

[3]This proposition resembles Theorem 1 in Pursiainen (2008, p. 8).

PROOF: For proving the necessity of (3) assume that A is consistent and define $\mathbf{d}_1 \oplus_A \mathbf{d}_2 = A_2(\mathbf{d}_1, \mathbf{d}_2)$. The symmetry of A_2 precisely means commutativity of $\oplus_A$. Associativity of $\oplus_A$ follows from

$$\begin{aligned}(\mathbf{d}_1 \oplus_A \mathbf{d}_2) \oplus_A \mathbf{d}_3 &= A_2((A_2(\mathbf{d}_1, \mathbf{d}_2), A_1(\mathbf{d}_3)) \\ &= A_3(\mathbf{d}_1, \mathbf{d}_2, \mathbf{d}_3) \\ &= A_2(A_1(\mathbf{d}_1), A_2(\mathbf{d}_2, \mathbf{d}_3)) \\ &= \mathbf{d}_1 \oplus_A (\mathbf{d}_2 \oplus_A \mathbf{d}_3) \ .\end{aligned}$$

The representation for the general case A_n is shown by induction on $n \in \mathbb{N}$. For $n = 1$, it follows from $A_1(\mathbf{d}_j) = \mathbf{d}_j$. If the necessity of (3) is true for some $n \in \mathbb{N}$ we get

$$\begin{aligned}A_{n+1}(\mathbf{d}_1, \ldots, \mathbf{d}_{n+1}) &= A_2((A_n(\mathbf{d}_1, \ldots, \mathbf{d}_n), A_1(\mathbf{d}_{n+1})) \\ &= A_n(\mathbf{d}_1, \ldots, \mathbf{d}_n) \oplus_A \mathbf{d}_{n+1} \\ &= (\mathbf{d}_1 \oplus_A \cdots \oplus_A \mathbf{d}_n) \oplus_A \mathbf{d}_{n+1} \ .\end{aligned}$$

It is obvious that the existence of a commutative and associative binary operation $\oplus_A$ that satisfies (3) is sufficient for $A = (A_n)_{n\in\mathbb{N}}$ to be a consistent aggregation rule. ■

Multiplication is a commutative, and associative binary operation $\oplus$ of form (3). This confirms that (2) is a consistent aggregation rule.

There is a simple but nevertheless quite general method to produce consistent aggregation rules or, conversely, to prove consistency of some given aggregation rule. It utilizes the concept of a "quasi-sum".

Definition 3. Let $M \subseteq \mathbb{R}^k$ be a set which is stable under addition (i.e., $\mathbf{m}_1 + \mathbf{m}_2 \in M$ for all $\mathbf{m}_1, \mathbf{m}_2 \in M$). If $\Phi : I \to M$ is any invertible map with inverse $\Phi^{-1} : M \to I$, we define a quasi-sum of $\mathbf{d}_1, \ldots, \mathbf{d}_n \in I$ by setting

$$\mathbf{d}_1 \oplus \cdots \oplus \mathbf{d}_n = \Phi^{-1}\left[\sum_{i=1}^{n} \Phi(\mathbf{d}_i)\right] \ . \tag{4}$$

Proposition 2. *In the situation of Definition 3 the aggregation rule*

$$A_n(\mathbf{d}_1, \ldots, \mathbf{d}_n) = \mathbf{d}_1 \oplus \cdots \oplus \mathbf{d}_n$$

is consistent.

Proof: In view of Proposition 1 it is to be shown that the binary operation $\oplus$ defined by (4) is commutative and associative. Commutativity of $\oplus$ is obvious. For $n = 3$ one gets

$$\begin{aligned}(\mathbf{d}_1 \oplus \mathbf{d}_2) \oplus \mathbf{d}_3 &= \Phi^{-1}\left[\Phi\left[\Phi^{-1}\left[\Phi(\mathbf{d}_1) + \Phi(\mathbf{d}_2)\right]\right] + \Phi(\mathbf{d}_3)\right] \\ &= \Phi^{-1}\left[\Phi(\mathbf{d}_1) + \Phi(\mathbf{d}_2) + \Phi(\mathbf{d}_3)\right] \\ &= \Phi^{-1}\left[\Phi(\mathbf{d}_1) + \Phi\left[\Phi^{-1}\left[\Phi(\mathbf{d}_2) + \Phi(\mathbf{d}_3)\right]\right]\right] \\ &= \mathbf{d}_1 \oplus (\mathbf{d}_2 \oplus \mathbf{d}_3)\ ,\end{aligned} \tag{5}$$

which is associativity. ■

It follows from Equation (5) that an aggregation rule A satisfies (4), if and only if

$$\Phi(A_n(\mathbf{d}_1, \ldots, \mathbf{d}_n)) = \sum_{i=1}^{n} \Phi(\mathbf{d}_i) \quad \text{for all } \ n \in \mathbb{N}\ . \tag{6}$$

An aggregation rule that satisfies Equation (6) is denoted here as *quasi-additive*.[4] Proposition 2 says that quasi-additive aggregation rules are consistent. In order to verify that a given aggregation rule A is consistent, it is therefore sufficient to find some invertible map $\Phi : I \to M$ such that Equation (6) holds for all $n \in \mathbb{N}$ and $\mathbf{d}_i \in I$.

Of course, the simple summation is additive and therefore also quasi-additive with $\Phi(d) = d$. As another example, consider aggregation rule (2). For $I = \mathbb{R}_{>0}$ it is consistent, because with $\Phi(d) = \log d$ Equation (6) is satisfied:

$$\log\left(\prod_{i=1}^{n} d_i\right) = \sum_{i=1}^{n} \log d_i\ .$$

We have emphasized the similarity between consistent aggregation rules and the sum or product of numbers or vectors. Aggregating a finite set of data to a kind of average or "typical value" is an alternative form of aggregation. A simple example is the arithmetic mean

$$A_n(d_1, \ldots, d_n) = \frac{1}{n}\sum_{i=1}^{n} d_i\ , \tag{7}$$

for $d_1, \ldots, d_n \in \mathbb{R}$. Although symmetric, this aggregation rule fails to be consistent, because it is not associative:

$$\frac{1}{3}(d_1 + d_2 + d_3) \neq \frac{1}{2}\left[\frac{1}{2}(d_1 + d_2) + d_3\right]\ ,$$

[4] There is quite some literature on the question which aggregation rules are quasi-additive. A good overview is given in the dissertation of Pursiainen (2005).

except for special cases.

However, the aggregation rule

$$A_n((d_1, w_1), \ldots, (d_n, w_n)) = \left(\left(\sum_{i=1}^{n} w_i \right)^{-1} \sum_{i=1}^{n} w_i d_i \ , \ \sum_{i=1}^{n} w_i \right) \tag{8}$$

overcomes this problem. By “inflating” the set I from $\mathbb{R}$ to $\mathbb{R} \times \mathbb{R}_{>0}$ the aggregation rule becomes a function of $\sum_{i=1}^{n} w_i d_i$ and $\sum_{i=1}^{n} w_i$. With $\Phi(\mathbf{d}) = \Phi(d, w) = (wd, w)$ we get

$$\begin{aligned}
\Phi(A_n(\mathbf{d}_1, \ldots, \mathbf{d}_n)) &= \Phi \left(\left(\sum_{i=1}^{n} w_i \right)^{-1} \sum_{i=1}^{n} w_i d_i \ , \ \sum_{i=1}^{n} w_i \right) \\
&= \left(\sum_{i=1}^{n} w_i \left(\sum_{i=1}^{n} w_i \right)^{-1} \sum_{i=1}^{n} w_i d_i \ , \ \sum_{i=1}^{n} w_i \right) \\
&= \left(\sum_{i=1}^{n} w_i d_i \ , \ \sum_{i=1}^{n} w_i \right) \\
&= \sum_{i=1}^{n} (w_i d_i, w_i) \\
&= \sum_{i=1}^{n} \Phi(\mathbf{d}_i) \ .
\end{aligned}$$

This verifies that aggregation rule (8) satisfies (6). Therefore, it is consistent. The relation between (7) and (8) is that the arithmetic mean (7) is the first component of $A_n((d_1, 1), \ldots, (d_n, 1))$. This shows how consistency is only achieved after augmenting the original aggregation rule.

Another simple example is the geometric mean

$$A_n(d_1, \ldots, d_n) = \left(\prod_{i=1}^{n} d_i \right)^{1/n} \ , \tag{9}$$

for $d_1, \ldots, d_n \in \mathbb{R}_{>0}$. Since

$$(d_1 d_2 d_3)^{1/3} \neq \left[(d_1 d_2)^{1/2} \, d_3 \right]^{1/2} \ ,$$

this aggregation rule is not associative. However, the aggregation rule

$$A_n((d_1, w_1), \ldots, (d_n, w_n)) = \left(\left(\prod_{i=1}^{n} d_i^{w_i} \right)^{1/\sum_{i=1}^{n} w_i} \ , \ \sum_{i=1}^{n} w_i \right) \tag{10}$$

is a function of $\prod_{i=1}^{n} d_i^{w_i}$ and $\sum_{i=1}^{n} w_i$. With $\Phi(\mathbf{d}) = \Phi(d, w) = (w \log d, w)$ we get

$$\begin{aligned}
\Phi(A_n(\mathbf{d}_1, \ldots, \mathbf{d}_n)) &= \Phi\left(\left(\prod_{i=1}^{n} d_i^{w_i}\right)^{1/\sum_{i=1}^{n} w_i}, \sum_{i=1}^{n} w_i\right) \\
&= \left(\left(\sum_{i=1}^{n} w_i\right)\log\left(\prod_{i=1}^{n} d_i^{w_i}\right)^{1/\sum_{i=1}^{n} w_i}, \sum_{i=1}^{n} w_i\right) \\
&= \left(\sum_{i=1}^{n} w_i \log d_i\ ,\ \sum_{i=1}^{n} w_i\right) \\
&= \sum_{i=1}^{n} (w_i \log d_i\ ,\ w_i) = \sum_{i=1}^{n} \Phi(\mathbf{d}_i)\ .
\end{aligned}$$

This establishes consistency of aggregation rule (10). Then, the geometric mean (9) is the first component of $A_n((d_1, 1), \ldots, (d_n, 1))$ and n is its second component.

4. Price Indices and Their Attributes

Price index formulae, too, are maps that compute some kind of average, namely the "overall price change". Therefore, one can attempt to transform them into a consistent aggregation rule. As a preliminary step, the concept of primary and secondary attributes of a price index must be introduced. The actual transformation of a price index into a consistent aggregation rule is deferred to Section 6.

Let S denote the set of integers $i = 1, \ldots, N$, where each integer represents one of the N items of an economy. All items are available during the base period ($t = 0$) and the comparison period ($t = 1$). The period t vector of prices is denoted by $\mathbf{p}^t = (p_1^t, \ldots, p_N^t)$, and the corresponding vector of quantities by $\mathbf{x}^t = (x_1^t, \ldots, x_N^t)$. It is customary to interpret a price index as a mapping of the N-dimensional vectors $\mathbf{p}^0$, $\mathbf{x}^0$, $\mathbf{p}^1$, and $\mathbf{x}^1$ into a single positive number, $P'(\mathbf{p}^0, \mathbf{x}^0, \mathbf{p}^1, \mathbf{x}^1)$.

In practical work, the prices and quantities p_i^0, x_i^0, p_i^1, and x_i^1 are usually not known. Instead, only the expenditures $v_i^0 = p_i^0 x_i^0$ and $v_i^1 = p_i^1 x_i^1$ as well as the price ratios $r_i = p_i^1 / p_i^0$ are available.[5] However, this does not represent a confinement as long as the applied price index formulae satisfy the commensurability axiom. This axiom postulates that

$$P'\left(\mathbf{p}^0\mathbf{\Lambda}, \mathbf{x}^0\mathbf{\Lambda}^{-1}, \mathbf{p}^1\mathbf{\Lambda}, \mathbf{x}^1\mathbf{\Lambda}^{-1}\right) = P'(\mathbf{p}^0, \mathbf{x}^0, \mathbf{p}^1, \mathbf{x}^1)\ ,$$

[5]In the price statistics literature, the variables v_i^0 and v_i^1 are usually denoted as "values". However, in the present paper the term "value"would have multiple meanings. To avoid confusion, we denote the variables v_i^0 and v_i^1 as "expenditures".

where $\mathbf{\Lambda}$ is some arbitrary $N \times N$ diagonal matrix with positive entries λ_i (e.g., Auer, 2004, pp. 386-387). For a price index that satisfies this axiom, the information in the four vectors $\mathbf{p}^0$, $\mathbf{x}^0$, $\mathbf{p}^1$, and $\mathbf{x}^1$ is equivalent to the information contained in the three vectors $\mathbf{r} = (r_1, \ldots, r_N)$, $\mathbf{v}^0 = (v_1^0, \ldots, v_N^0)$, and $\mathbf{v}^1 = (v_1^1, \ldots, v_N^1)$. Therefore, we get $P'(\mathbf{p}^0, \mathbf{x}^0, \mathbf{p}^1, \mathbf{x}^1) = P'(\mathbf{1}, \mathbf{v}^0, \mathbf{r}, \mathbf{v}^1/\mathbf{r}) = P(\mathbf{r}, \mathbf{v}^0, \mathbf{v}^1)$, where $\mathbf{1} = (1, \ldots, 1)$. This is the commensurability for $\lambda_i = 1/p_i^0$.

As an example, consider the Walsh index:

$$P'^{\text{Wa}}(\mathbf{p}^0, \mathbf{x}^0, \mathbf{p}^1, \mathbf{x}^1) = \frac{\sum_{i \in S} p_i^1 \sqrt{x_i^0 x_i^1}}{\sum_{i \in S} p_i^0 \sqrt{x_i^0 x_i^1}} .$$

This price index satisfies the commensurability axiom. Therefore, it can be written in the following form:

$$P^{\text{Wa}}(\mathbf{r}, \mathbf{v}^0, \mathbf{v}^1) = \sum_{i \in S} r_i \frac{\sqrt{v_i^0 v_i^1 / r_i}}{\sum_{j \in S} \sqrt{v_j^0 v_j^1 / r_j}} . \tag{11}$$

Just like the Laspeyres index,

$$P^{\text{La}}(\mathbf{r}, \mathbf{v}^0, \mathbf{v}^1) = \sum_{i \in S} r_i \frac{v_i^0}{\sum_{j \in S} v_j^0} ,$$

the Walsh index can be interpreted as a weighted arithmetic mean of the price ratios, r_i, where the weights represent expenditure weights. However, whereas the Laspeyres weights use only base period expenditures, v_i^0, the weights of the Walsh index are geometric means of the base period expenditures, v_i^0, and the "deflated" comparison period expenditures, v_i^1/r_i.

It is well known that all sensible price index formulae satisfy the commensurability axiom (e.g., Auer, 2004, p. 393). Therefore, any sensible price index can be written in the form $P(\mathbf{r}, \mathbf{v}^0, \mathbf{v}^1)$ and we can take $J = \mathbb{R}^3_{>0}$ as the set of possible data, (r_i, v_i^0, v_i^1), for every item i.

It is customary to interpret a price index formula as a rule that aggregates a finite set of data $(\mathbf{r}, \mathbf{v}^0, \mathbf{v}^1) \in J^N$ into a single datum $P(\mathbf{r}, \mathbf{v}^0, \mathbf{v}^1) \in \mathbb{R}_{>0}$, where it is understood that for each size of the data set, this mapping has the "same functional form". As pointed out before, the expression "same functional form" is not quite appropriate, because these mappings have different domains, and therefore, are totally different objects. As a consequence, the customary definition of a price index formula is not fully satisfactory. To account for different domains, we introduce the following definition of a price index (see also Definition 3.1 in Pursiainen, 2005, p. 21).

Definition 4. A price index P for $J = \mathbb{R}^3_{>0}$ is a sequence $P = (P_n)_{n\in\mathbb{N}}$ of symmetric and continuous maps

$$P_n : J^n \to \mathbb{R}_{>0}\,, \qquad (\mathbf{r}, \mathbf{v}^0, \mathbf{v}^1) \longmapsto P_n(\mathbf{r}, \mathbf{v}^0, \mathbf{v}^1)\,,$$

with $P_1(r_i, v_i^0, v_i^1) = r_i$ for all $(r_i, v_i^0, v_i^1) \in J$.

The primary purpose of a price index formula is the computation of the overall price change P of the items in set S, given the data set $(\mathbf{r}, \mathbf{v}^0, \mathbf{v}^1) \in J^N$. Restricting the set S to a single item i, the "overall price change" should be the item's price ratio, $r_i = p_i^1/p_i^0$. Therefore, the price ratio r_i is denoted here as the *primary attribute* of a price index. A price index can be interpreted as a transformation of the primary attribute's values $\mathbf{r}$ into some aggregate value P_n. However, the value of P_n depends not only on $\mathbf{r}$, but also on $\mathbf{v}^0$ and $\mathbf{v}^1$. Therefore, we denote v_i^0 and v_i^1 as *secondary attributes*. More generally, secondary attributes are defined in the following way:

Definition 5. A vector valued secondary attribute is a mapping

$$\mathbf{z} = \left(z^1, \ldots, z^Q\right) : J \to M \subseteq \mathbb{R}^Q_{>0}\,,$$

where $Q \geq 1$ and $J = \mathbb{R}^3_{>0}$ is the set containing the original data (r, v^0, v^1).

This definition implies that a secondary attribute's value corresponding to some item i, z_i^q $(q = 1, \ldots, Q)$, exclusively depends on (r_i, v_i^0, v_i^1). A price index can have alternative vectors of secondary attributes. For example, the Walsh Index (11) is a function of the primary attribute, r_i, and the secondary attribute $\mathbf{z}_i = \left(z_i^1, z_i^2\right) = \left(v_i^0, v_i^1\right)$. However, this index can be written also with a single secondary attribute: $\mathbf{z}_i = \sqrt{v_i^0 v_i^1 / r_i}$.

Before we move on to develop a rigorous definition of consistency in aggregation of price indices, we present an empirical application of a two stage price index computation.

5. Application to Swedish Price Data: Part A

The underlying data of this empirical application have been acquired from Statistics Sweden. They cover the base year 2010 ($t = 0$) and the comparison year 2011 ($t = 1$). The informational set is $(\mathbf{r}, \mathbf{v}^0, \mathbf{v}^1) \in J^{360}$ where the elements v_i^0 and v_i^1 are annual household expenditures on 360 basic headings i and the elements r_i are the respective price ratios. As pointed out before, for the purpose of computing a price index, this informational set is as good as the set $\left(\mathbf{p}^0, \mathbf{x}^0, \mathbf{p}^1, \mathbf{x}^1\right)$.

Both the expenditure and price data are disaggregated at the four-digit level COICOP classification. Table 1 shows an excerpt of the original data set. It lists for each basic heading i the COICOP number, the product group number, the product name, the price ratio r_i, and the expenditures v_i^0 and v_i^1. The last column can be ignored for the moment.

Table 1: Numerical Illustration* - Two-Stage Aggregation of Walsh Index

	Basic Heading Information							Sec. Attrib.
	i	Coicop	Group	Product	r_i	v_i^0	v_i^1	$z_i = \sqrt{v_i^0 v_i^1 / r_i}$
Core inflation (S_1)	1	01.1.1	1113	Wheat Bread	1.0333	1524	1562	1517.8
	2	01.1.1	1114	Danish Pastry	1.0318	203	208	202.3
	3	01.1.1	1116	Cookies	1.0131	664	676	665.6
	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
	76	03.1	3206	Men Jacket	1.0774	3505	3571	3408.4
	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
	301	12.7	9704	Lawyer Fees	1.0282	1067	1085	1061.1
				$P_1^{\text{Wa}} = 1.0264$				$Z_1 = 1\,257\,744.8$
Non-Core inflation (S_2)	302	01.1.3	1307	Herring	1.0438	155	128	137.9
	303	01.1.3	1314	Cod	0.9234	272	164	219.8
	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
	312	01.1.6	1617	Pears	0.9446	501	469	498.7
	313	01.1.6	1618	Apples	1.0455	1354	1775	1516.2
	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
	356	04.5.x	4702	Fuel Oil	1.1278	2150	1765	1834.3
	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
	360	07.2.2	6225	E 85 Fuel	1.0479	1205	1245	1196.5
				$P_2^{\text{Wa}} = 1.0355$				$Z_2 = 178\,317.6$

∗ Source: Statistics Sweden, Consumer Price Index Data for 2010-2011.

In 2005, Statistics Sweden implemented the Walsh index (11) for the compilation of its consumer price index (for details see Bäckström and Sammar, 2012, p. 2). Therefore, the same price index is used here. A single stage computation of the Walsh index

(11) yields the index number $P^{\mathrm{Wa}} = 1.0275$, that is, an overall inflation of 2.75%.[6]

For the two stage computation, the 360 items are partitioned into the two subsets S_1 (core inflation) and S_2 (non-core inflation), where the items $i = 1, 2, .., 301$ are assigned to the subset S_1 while the items $i = 302, \ldots, 360$ are assigned to the subset S_2. An item's price ratio, r_i, is defined as its primary attribute and the term $z_i = \sqrt{v_i^0 v_i^1 / r_i}$ is chosen as the only secondary attribute of the Walsh index (therefore, at z_i no superscript is necessary). For each item i the value of its primary attribute is listed in Table 1 in the column with the heading r_i while the value of its secondary attribute $z_i = \sqrt{v_i^0 v_i^1 / r_i}$ is listed in the last column.

For each subset the value of its primary attribute (P_1^{Wa} and P_2^{Wa}) is computed by the Walsh index formula

$$P_k^{\mathrm{Wa}} = \sum_{i \in S_k} r_i \frac{z_i}{\sum_{j \in A_k} z_j} . \tag{12}$$

This yields $P_1^{\mathrm{Wa}} = 1.0264$ and $P_2^{\mathrm{Wa}} = 1.0355$. In other words, the Swedish core inflation is 2.64%, while the non-core inflation is 3.55%. Recall that the overall inflation rate was 2.75%, that is, slightly larger than the core inflation rate. Note that formula (12) is the same as (11), the formula applied for the single stage computation. The aggregate values of the secondary attributes are

$$Z_1 = \sum_{i \in S_1} z_i = 1\,257\,744.8 \qquad \text{and} \qquad Z_2 = \sum_{i \in S_2} z_i = 178\,317.6 .$$

These two numbers are also listed in Table 1.

The second stage index formula is

$$P^{\mathrm{Wa}} = \sum_{k=1}^{K} P_k^{\mathrm{Wa}} \frac{Z_k}{\sum_{l=1}^{K} Z_l} , \tag{13}$$

with $K = 2$. This is again the same basic formula as in the single stage computation. Inserting the results of the first stage computations $\left(P_1^{\mathrm{Wa}}, Z_1\right)$ and $\left(P_2^{\mathrm{Wa}}, Z_2\right)$ in the second stage formula (13) yields the two stage index number $P^{\mathrm{Wa}} = 1.0275$. This is exactly the same index number as in the single stage computation (12). Furthermore, the Walsh index (12) seems to satisfy all of the four consensus conditions. This suggests that the Walsh index might be consistent in aggregation.

This is a remarkable conjecture, because "... something resembling a consensus has emerged in the index number literature that inflation and growth should be measured using superlative index number formulae ... (Hill, 2006, p. 27)". The Walsh index is

[6]Our data and our price index formulae are not completely equivalent to the data and methodology underlying the compilation of the official Swedish consumer price index.

one of the three advocated superlative price indices, the others being the Fisher index,

$$P^{\mathrm{Fi}} = \left(\frac{\sum_{i \in S} v_i^0 r_i}{\sum_{i \in S} v_i^0} \frac{\sum_{i \in S} v_i^1}{\sum_{i \in S} v_i^1 / r_i} \right)^{1/2} , \tag{14}$$

and the Törnqvist index,

$$\ln P^{\mathrm{Tö}} = \sum_{i \in S} \ln (r_i) \frac{1}{2} \left(\frac{v_i^0}{\sum_{j \in S} v_j^0} + \frac{v_i^1}{\sum_{j \in S} v_j^1} \right) . \tag{15}$$

The concept of superlative price indices was introduced by Diewert (1976). A price index receives the title "superlative", if an aggregator function (utility function or expenditure function) with a "flexible" functional form exists, such that its corresponding cost of living index yields the same result as the price index. An aggregator function is "flexible", if it can provide a second-order approximation to an arbitrary twice differentiable linearly homogeneous aggregator function.

It is well known that the superlative indices of Walsh, Fisher, and Törnqvist possess a number of desirable properties and that they approximate each other closely (e.g., Hill, 2006, p. 27). However, there is a general perception that none of these superlative price indices is consistent in aggregation (e.g., van Yzeren, 1958, p. 432-433; Diewert, 1978, p. 889; Diewert, 2004a, p. 349-350; Auer, 2004, p. 397; Balk, 2008, pp. 107-108). Even though Diewert (1978, p. 889) shows that these indices are "approximately consistent in aggregation", empirical studies that conduct a multi stage analysis usually avoid superlative price indices. Apparently, a superlative index that is merely approximately consistent in aggregation, is not considered as suitable for a multi stage analysis. The empirical application of the present paper suggests that, contrary to general perception, at least one of the (superlative) indices is consistent in aggregation. In the remaining sections, this conjecture is verified.

6. Consistent Aggregation of Price Indices

A price index P in the sense of Definition 4 is not an aggregation rule in the sense of Definition 1, because the maps P_n have values in $\mathbb{R}_{>0}$ instead of $J = \mathbb{R}^3_{>0}$. However, a price index formula P can be transformed to become an aggregation rule in the sense of Definition 1. The first step is to transform the original data set: $\left(r_i, v_i^0, v_i^1\right) \longmapsto \mathbf{d}_i = (r_i, \mathbf{z}_i) \in I = \mathbb{R}_{>0} \times M \subseteq \mathbb{R}^Q_{>0}$, $i = 1, \ldots, n$, where $\mathbf{z} : J \rightarrow M$ is a secondary attribute and $\mathbf{z}_i = \mathbf{z}\left(r_i, v_i^0, v_i^1\right)$. In a second step, an aggregation rule must be specified in the sense of Definition 1, $A_n(\mathbf{d}_1, \ldots, \mathbf{d}_n)$.

For example, the Walsh index (11) can be transformed into an aggregation rule $A_n^{\mathrm{Wa}}(\mathbf{d}_1, \ldots, \mathbf{d}_n)$ with $\mathbf{d}_i = (r_i, \mathbf{z}_i)$ and $\mathbf{z}_i = z_i = \sqrt{v_i^0 v_i^1 / r_i}$. This aggregation rule has two components. The first one is the price index formula (11). It maps the data set I^n into $\mathbb{R}_{>0}$, that is, into some aggregated value of the primary attribute. This aggregate value depends on the individual values of the primary and secondary attribute. The aggregation rule's second component is a mapping that transforms the individual values of the secondary attribute z_i into some aggregate value $\sum_{i=1}^{n} z_i$. This aggregate value exclusively depends on the individual values of the secondary attribute. Finally, the two components are combined to

$$A_n^{\mathrm{Wa}}(\mathbf{d}_1, \ldots, \mathbf{d}_n) = \left(\sum_{i=1}^{n} r_i \frac{z_i}{\sum_{j=1}^{n} z_j}\ ,\ \sum_{i=1}^{n} z_i \right), \tag{16}$$

with $\mathbf{d}_i = (r_i, \mathbf{z}_i) = (r_i, z_i) = \left(r_i, \sqrt{v_i^0 v_i^1 / r_i}\right)$. The maps A_n^{Wa} form an aggregation rule A for $I = \mathbb{R}_{>0}^2$ in the sense of Definition 1. The superscript "Wa" emphasizes that this aggregation rule corresponds to the Walsh Index.

As a second example, consider the Fisher index (14) and define the secondary attribute $\mathbf{z}_i = \left(z_i^1, z_i^2, z_i^3, z_i^4\right) = \left(v_i^0, v_i^1, v_i^0 r_i, v_i^1 / r_i\right)$. The Fisher index P^{Fi} can be transformed into the aggregation rule

$$A_n^{\mathrm{Fi}}(\mathbf{d}_1, \ldots, \mathbf{d}_n) = \left(\left(\frac{\sum_{i=1}^{n} z_i^3}{\sum_{i=1}^{n} z_i^1} \frac{\sum_{i=1}^{n} z_i^2}{\sum_{i=1}^{n} z_i^4} \right)^{1/2}, \sum_{i=1}^{n} z_i^1, \sum_{i=1}^{n} z_i^2, \sum_{i=1}^{n} z_i^3, \sum_{i=1}^{n} z_i^4 \right), \tag{17}$$

with $\mathbf{d}_i = (r_i, \mathbf{z}_i) = \left(r_i, z_i^1, z_i^2, z_i^3, z_i^4\right) = \left(r_i, v_i^0, v_i^1, v_i^0 r_i, v_i^1 / r_i\right)$.[7] This aggregation rule has five components. The first one is the price index formula (14). In contrast to the first component of A_n^{Wa}, the first component of A_n^{Fi} does not depend on the values of the primary attribute, but only on the values of the secondary attribute $\mathbf{z}_i$.

In Definitions 4 and 5 a price index P and its secondary attribute $\mathbf{z}$ were defined. Consistency of an aggregation rule was defined in Definition 2. Building on these three Definitions, we define consistency of aggregation of a price index formula P with an explicit reference to its secondary attribute:

Definition 6. A price index $P = (P_n)_{n \in \mathbb{N}}$ is consistent in aggregation with respect to a secondary attribute $\mathbf{z} : J \to M$, if there is a consistent aggregation rule $A = (A_n)_{n \in \mathbb{N}}$ for $I = \mathbb{R}_{>0} \times M$ with continuous A_n such that $P_n(\mathbf{r}, \mathbf{v}^0, \mathbf{v}^1)$ with $(\mathbf{r}, \mathbf{v}^0, \mathbf{v}^1) \in J^n$ is the first component of $A_n(\mathbf{d}_1, \ldots, \mathbf{d}_n)$ for all $n \in \mathbb{N}$ and $\mathbf{d}_i = (r_i, \mathbf{z}_i) \in I$ with $i = 1, \ldots, n$, where $\mathbf{z}_i = \mathbf{z}\left(r_i, v_i^0, v_i^1\right)$.

This definition emphasizes the continuity of the consistent aggregation rule A. In

[7] We owe this insight to Bjørn Kjos-Hanssen.

the Appendix (Proposition 7) it is shown that a neglect of continuity has absurd consequences.

In Definition 6 the phrase "with respect to a secondary attribute $\mathbf{z}$" is important, because the definition allows for a wide range of possible secondary attributes and not all of them may appear appealing. Different views on what constitutes an admissable secondary attribute have given rise to a wide variety of definitions of consistency in aggregation (discussed in Section 10). However, a discussion of what constitutes an admissable secondary attribute, is deferred to Section 10. For the time being, we examine whether price indices exist that are consistent in aggregation with respect to some secondary attribute $\mathbf{z}$ in the broad sense of Definition 6. We begin with the three superlative price indices (Fisher, Törnqvist, Walsh).

7. Consistency of Superlative Price Indices

To prove that some price index is consistent in aggregation with respect to some $\mathbf{z}$ one has to transform the price index in an aggregation rule that is consistent. For example, consider again the weighted arithmetic mean defined by (8). We know that this aggregation rule is consistent. The transformed Walsh index (16) is a special case of (8) with $d_i = r_i$ and $w_i = z_i = \sqrt{v_i^0 v_i^1 / r_i}$. The first component of (16) is the Walsh index $P_n^{\mathrm{Wa}}(\mathbf{r}, \mathbf{v}^0, \mathbf{v}^1)$. Therefore, the Walsh index is consistent in aggregation with respect to the secondary attribute $\mathbf{z}_i = \sqrt{v_i^0 v_i^1 / r_i}$.

For some aggregation rules, Proposition 2 provides an elegant route to prove the rule's consistency. A similar route exists for proving that a price index is consistent in aggregation with respect to some secondary attribute $\mathbf{z}$.

Proposition 3. *Let $P = (P_n)_{n\in\mathbb{N}}$ be a price index with a secondary attribute $\mathbf{z} : J \to M \subseteq \mathbb{R}^Q_{>0}$, where M is stable under addition. Set $I = \mathbb{R}_{>0} \times M$ and assume that there is a continuous function $f : I \to L$ (where L is either $\mathbb{R}_{>0}$ or $\mathbb{R}$) such that, for each $\mathbf{m} \in M$, the partial function $r \mapsto f(r, \mathbf{m})$ is invertible on $\mathbb{R}_{>0}$ and that*

$$f\left(P_n\left(\mathbf{r}, \mathbf{v}^0, \mathbf{v}^1\right)\,,\ \sum_{i=1}^{n} \mathbf{z}_i\right) = \sum_{i=1}^{n} f(r_i, \mathbf{z}_i) \tag{18}$$

for all $n \in \mathbb{N}$, $(\mathbf{r}, \mathbf{v^0}, \mathbf{v^1}) = (r_i, v_i^0, v_i^1)_{i\leq n} \in J^n$, and $\mathbf{z}_i = \mathbf{z}\left(r_i, v_i^0, v_i^1\right)$. Then P is consistent in aggregation with respect to $\mathbf{z}$.

Proof: We define $\Phi : I \to L \times M$ by $\Phi(r, \mathbf{m}) = (f(r, \mathbf{m}), \mathbf{m})$. The invertibility of $r \mapsto f(r, \mathbf{m})$ implies that Φ is also invertible. For $\mathbf{d}_i = (r_i, \mathbf{m}_i)$, we can define the quasi-sum

$$\mathbf{d}_1 \oplus_A \mathbf{d}_2 = \Phi^{-1}(\Phi(\mathbf{d}_1) + \Phi(\mathbf{d}_2)) \ .$$

It follows from Proposition 2 that $A_n(\mathbf{d}_1, \ldots, \mathbf{d}_n) = \mathbf{d}_1 \oplus_A \cdots \oplus_A \mathbf{d}_n$ is a consistent aggregation rule for I. The continuity of f implies that of A_n. It is left to be shown that $P_n\left(\mathbf{r}, \mathbf{v}^0, \mathbf{v}^1\right)$ is the first component of $A_n(\mathbf{d}_1, \ldots, \mathbf{d}_n)$. Using the definition of Φ and the assumptions on f, we have

$$\begin{aligned}
\Phi\left(P_n\left(\mathbf{r}, \mathbf{v}^0, \mathbf{v}^1\right) \ , \ \sum_{i=1}^n \mathbf{z}_i\right) &= \left(f\left(P_n\left(\mathbf{r}, \mathbf{v}^0, \mathbf{v}^1\right) \ , \ \sum_{i=1}^n \mathbf{z}_i\right) \ , \ \sum_{i=1}^n \mathbf{z}_i\right) \\
&= \left(\sum_{i=1}^n f(r_i, \mathbf{z}_i) \ , \ \sum_{i=1}^n \mathbf{z}_i\right) \\
&= \sum_{i=1}^n (f(r_i, \mathbf{z}_i) \ , \ \mathbf{z}_i) \\
&= \sum_{i=1}^n \Phi(\mathbf{d}_i) \ .
\end{aligned}$$

This gives

$$\begin{aligned}
\left(P_n\left(\mathbf{r}, \mathbf{v}^0, \mathbf{v}^1\right) \ , \ \sum_{i=1}^n \mathbf{z}_i\right) &= \Phi^{-1}\left(\sum_{i=1}^n \Phi(\mathbf{d}_i)\right) \\
&= \mathbf{d}_1 \oplus_A \cdots \oplus_A \mathbf{d}_n \\
&= A_n(\mathbf{d}_1, \ldots, \mathbf{d}_n) \ ,
\end{aligned}$$

as required. ■

Utilizing Proposition 3, it is easy to show the first part of the following result:

Proposition 4. *The Walsh index (11) is consistent in aggregation with respect to the secondary attribute* $\mathbf{z} = \sqrt{v^0 v^1 / r}$. *The Fisher index (14) is consistent in aggregation with respect to the secondary attribute* $\mathbf{z} = \left(v^0, v^1, v^0 r, v^1 / r\right)$.

Proof: Multiplying both sides of Equation (11) by $\sum_{j=1}^N \sqrt{v_j^0 v_j^1 / r_j}$, the Walsh index, P_n^{Wa}, can be expressed as in Equation (18) with

$$f\left(P_n^{\text{Wa}}(\mathbf{d}_1, \ldots, \mathbf{d}_n) \ , \ \sum_{i=1}^n \mathbf{z}_i\right) = P_N^{\text{Wa}}(\mathbf{d}_1, \ldots, \mathbf{d}_n) \sum_{i=1}^N \sqrt{v_i^0 v_i^1 / r_i}$$

and

$$f(\mathbf{d}_i) = f\left(r_i, \sqrt{v_i^0 v_i^1 / r_i}\right) = r_i \sqrt{v_i^0 v_i^1 / r_i} \ .$$

The function f satisfies the assumptions stated in Proposition 3. For the Fisher index, P_n^{Fi}, it suffices to point out that aggregation rule (17) is consistent and satisfies the properties stated in Definition 6. ■

The Törnqvist index (15) is another prominent superlative price index. It turns out that this index is consistent in aggregation with respect to the secondary attribute $\mathbf{z} = (r, v^0, r, v^1)$. To show this, we write this index in the following form:

$$P^{\mathrm{Tö}} = \sqrt{\widehat{P}\widetilde{P}}$$

with

$$\ln \widehat{P} = \sum_{i \in S} \ln(r_i) \frac{v_i^0}{\sum_{j \in S} v_j^0} \qquad \text{and} \qquad \ln \widetilde{P} = \sum_{i \in S} \ln(r_i) \frac{v_i^1}{\sum_{j \in S} v_j^1} .$$

These two simpler indices are consistent in aggregation with respect to $\mathbf{z} = v^0$ and $\mathbf{z} = v^1$, respectively (as is easily seen by Proposition 3 with $f(\mathbf{d}_i) = z_i \ln r_i$). Consistency in aggregation of the Törnqvist index is thus implied by the following:

Proposition 5. *Let $\widehat{P}$ and $\widetilde{P}$ be two price indices which are consistent in aggregation with respect to the secondary attributes $\widehat{\mathbf{z}} : J \to \widehat{M}$ and $\widetilde{\mathbf{z}} : J \to \widetilde{M}$ and corresponding aggregation rules $\widehat{A}_n(\widehat{\mathbf{d}}_1, \ldots, \widehat{\mathbf{d}}_n) = \widehat{\mathbf{d}}_1 \widehat{\oplus} \cdots \widehat{\oplus} \widehat{\mathbf{d}}_n$ and $\widetilde{A}_n(\widetilde{\mathbf{d}}_1, \ldots, \widetilde{\mathbf{d}}_n) = \widetilde{\mathbf{d}}_1 \widetilde{\oplus} \cdots \widetilde{\oplus} \widetilde{\mathbf{d}}_n$. Then the geometric mean $P_n = \sqrt{\widehat{P}_n \widetilde{P}_n}$ and the arithmetic mean $\overline{P}_n = \left(\widehat{P}_n + \widetilde{P}_n\right)/2$ are consistent in aggregation with respect to $\mathbf{z}(r, v^0, v^1) = \left(r, \widehat{\mathbf{z}}(r, v^0, v^1), r, \widetilde{\mathbf{z}}(r, v^0, v^1)\right)$.*

Proof: We show the statement for the geometric mean. Given

$$\mathbf{d}_j = \left(r_j, \widehat{x}_j, \widehat{\mathbf{m}}_j, \widetilde{x}_j, \widetilde{\mathbf{m}}_j\right) \in \hat{I} = \mathbb{R}_{>0} \times \mathbb{R}_{>0} \times \widehat{M} \times \mathbb{R}_{>0} \times \widetilde{M},$$

we set

$$\widehat{\mathbf{a}} = (\widehat{x}_1, \widehat{\mathbf{m}}_1) \widehat{\oplus} (\widehat{x}_2, \widehat{\mathbf{m}}_2) \qquad \text{and} \qquad \widetilde{\mathbf{a}} = (\widetilde{x}_1, \widetilde{\mathbf{m}}_1) \widetilde{\oplus} (\widetilde{x}_2, \widetilde{\mathbf{m}}_2)$$

and define

$$\mathbf{d}_1 \oplus \mathbf{d}_2 = \left(\sqrt{\pi_1(\widehat{\mathbf{a}}) \pi_1(\widetilde{\mathbf{a}})}, \widehat{\mathbf{a}}, \widetilde{\mathbf{a}} \right) ,$$

where $\pi_1(\cdot)$ denotes the first component of a vector. The aggregation rule $A_n(\mathbf{d}_1, \ldots, \mathbf{d}_n) = \mathbf{d}_1 \oplus \cdots \oplus \mathbf{d}_n$ is consistent by Proposition 1, since commutativity and associativity of $\widehat{\oplus}$ and $\widetilde{\oplus}$ transfer to $\oplus$. Moreover, it is clear that

$\sqrt{\widehat{P}_n\left((r_i, v_i^0, v_i^1)_{i\leq n}\right)\widetilde{P}\left((r_i, v_i^0, v_i^1)_{i\leq n}\right)}$ is the first component of $A_n(\mathbf{d}_1, \ldots, \mathbf{d}_n)$ for $\mathbf{d}_i = \left(r_i, \mathbf{z}(r_i, v_i^0, v_i^1)\right)$. ■

Applying this proof to the Törnqvist index, the transformed data $\mathbf{d}_i = (r_i, r_i, v_i^0, r_i, v_i^1)$ are aggregated according to

$$\mathbf{d}_1 \oplus \cdots \oplus \mathbf{d}_n = \left(P_n, \widehat{P}_n, \sum_{i=1}^{n} v_i^0, \widetilde{P}_n, \sum_{i=1}^{n} v_i^1\right) .$$

The Törnqvist index is consistent in aggregation with respect to the secondary attribute $\mathbf{z} = (r, v^0, r, v^1)$. In this case the price ratios, r_i, represent not only the primary attribute, but at the same time two secondary attributes. This reinforces a point that we made before: Definition 6 leaves a lot of, perhaps too much, scope of discretion in our choice of secondary attributes. In Section 10 we will return to this issue. There we discuss additional requirements that restrict the set of admissable secondary attributes and the way these attributes are aggregated.

8. Consistency of Other Price Indices

Besides the superlative price indices of Fisher, Törnqvist, and Walsh, numerous other price indices exist that are consistent in aggregation with respect to some secondary attribute $\mathbf{z}$.

Proposition 6. *The price index formulae P listed in Tables 2 and 3 are consistent in aggregation with respect to the secondary attributes specified in the last column of these tables.*

Proof: Tables 2 and 3 list for each price index the corresponding function $f(r_i, \mathbf{z}_i)$ and the secondary attributes z_i^q. Via Proposition 3, the function $f(r_i, \mathbf{z}_i)$ yields an explicit construction for the aggregation rule A required in the definition of consistency. ■

Table 2 lists a number of traditional price indices, whereas many indices that Auer (2014) categorizes as generalized unit value indices are listed in Table 3.

Table 2: Traditional Price Indices and Their Secondary Attributes

Name	Price Index Formula	Function $f(r_i, z_i^1, ..., z_i^Q)$	Secondary Attributes z_i^q
Laspeyres	$P^{\text{La}} = \frac{\sum v_i^0 r_i}{\sum v_i^0}$	$r_i\, z_i$	v_i^0
Paasche	$P^{\text{Pa}} = \frac{\sum v_i^1}{\sum v_i^1 / r_i}$	$r_i^{-1}\, z_i$	v_i^1
Marshall-Edgeworth	$P^{\text{ME}} = \sum r_i \frac{v_i^0 + v_i^1/r_i}{\sum \left(v_i^0 + v_i^1/r_i\right)}$	$r_i\, z_i$	$(v_i^0 + v_i^1/r_i)$
Walsh-2	$\ln P^{\text{Wa2}} = \sum \ln r_i \frac{\sqrt{v_i^0 v_i^1}}{\sum \sqrt{v_j^0 v_j^1}}$	$\ln(r_i)\, z_i$	$\sqrt{v_i^0 v_i^1}$
Walsh-Vartia	$\ln P^{\text{WV}} = \sum \ln r_i \frac{\sqrt{v_i^0}}{\sqrt{\sum v_j^0}} \frac{\sqrt{v_i^1}}{\sqrt{\sum v_j^1}}$	$\ln(r_i) \sqrt{z_i^1 z_i^2}$	$v_i^0,\ v_i^1$
Theil	$\ln P^{\text{Th}} = \sum \ln r_i \frac{\sqrt[3]{\frac{1}{2}(v_i^0 + v_i^1) v_i^0 v_i^1}}{\sum \sqrt[3]{\frac{1}{2}(v_j^0 + v_j^1) v_j^0 v_j^1}}$	$\ln(r_i)\, z_i$	$\sqrt[3]{\frac{1}{2}(v_i^0 + v_i^1) v_i^0 v_i^1}$
Vartia*	$\ln P^{\text{Va}} = \sum \ln r_i \frac{L(v_i^0, v_i^1)}{L(\sum v_i^0, \sum v_i^1)}$ with $L(a,b) = \begin{cases} \frac{b-a}{\ln b - \ln a} & \text{for } a \neq b \\ a & \text{for } a = b \end{cases}$	$\ln(r_i)\, L(z_i^1, z_i^2)$	$v_i^0,\ v_i^1$

∗ See Vartia (1976b, pp. 122-123). The index is sometimes called the Montgomery-Vartia index (e.g., Balk, 2008, p. 87).

Table 3: Generalized Unit Value Indices and Their Secondary Attributes

Name	Price Index Formula	Function $f(r_i, z_i^1, ..., z_i^Q)$	Secondary Attributes z_i^q
Banerjee (GUV-3)**	$P^{\text{Ba}} = \frac{\sum v_i^1}{\sum v_i^0} \frac{\sum v_i^0 (1 + r_i)}{\sum v_i^1 (1 + 1/r_i)}$	$r_i \; \frac{z_i^1}{z_i^2} \; z_i^3$	$v_i^0, v_i^1, v_i^1 \frac{1 + r_i}{r_i}$
Davies (GUV-4)**	$P^{\text{Da}} = \frac{\sum v_i^1}{\sum v_i^0} \frac{\sum v_i^0 \sqrt{r_i}}{\sum v_i^1 \sqrt{1/r_i}}$	$r_i \; \frac{z_i^1}{z_i^2} \; z_i^3$	$v_i^0, v_i^1, v_i^1 / \sqrt{r_i}$
(GUV-5)**	$P^{\text{GUV-5}} = \frac{\sum v_i^1}{\sum v_i^0} \frac{\sum v_i^0 \left(1 + r_i^{-1}\right)^{-1}}{\sum v_i^1 (1 + r_i)^{-1}}$	$r_i \; \frac{z_i^1}{z_i^2} \; z_i^3$	$v_i^0, v_i^1, v_i^1 / (r_i + 1)$
(GUV-6)**	$P^{\text{GUV-6}} = \frac{\sum v_i^1}{\sum v_i^0} \frac{\sum v_i^0 r_i^{v_i^1/(v_i^0 + v_i^1)}}{\sum v_i^1 r_i^{-v_i^0/(v_i^0 + v_i^1)}}$	$r_i \; \frac{z_i^1}{z_i^2} \; z_i^3$	$v_i^0, v_i^1, v_i^1 r_i^{\frac{-v_i^0}{v_i^0 + v_i^1}}$
Lehr (GUV-7)**	$P^{\text{Le}} = \frac{\sum v_i^1}{\sum v_i^0} \frac{\sum v_i^0 \left(v_i^0 + v_i^1\right)\left(v_i^0 + v_i^1 / r\right)}{\sum v_i^1 \left(v_i^0 + v_i^1\right)\left(v_i^0 r_i + v_i^1\right)}$	$r_i \; \frac{z_i^1}{z_i^2} \; z_i^3$	$v_i^0, v_i^1, v_i^1 \frac{v_i^0 + v_i^1}{r_i v_i^0 + v_i^1}$

** See Auer (2014, pp. 850-52).

9. Application to Swedish Price Data: Part B

The three superlative price indices and all price indices listed in Tables 2 and 3 have been applied to the Swedish data set described in Section 5. Table 4 reports the Swedish overall, core, and non-core inflation rates as measured by the various price indices. In the last four columns the table shows the aggregate values of the respective secondary attributes.

The various index formulae produce very similar results for the overall inflation (second column). The same is true for the core inflation as well as for the non-core inflation (fourth column). As expected, the largest values are produced by the Laspeyres index, whereas the Paasche index generates the smallest values.

Table 4: Numerical Illustration* - Comparison of One- and Two Stage Aggregation

Name	One-Stage Aggregation	Two-Stage Aggregation					
				Secondary Attributes Z_k^q			
		k	P_k	Z_k^1	Z_k^2	Z_k^3	Z_k^4
Fisher	1.027496	1	1.026370	1 243 742	1 306 914	1 277 026	1 273 822
		2	1.035407	182 775	180 275	189 468	174 315
Törnqvist	1.027699	1	1.026586	1.023309	1 243 742	1.029874	1 306 914
		2	1.035481	1.034459	182 775	1.036504	180 275
Walsh	1.027516	1	1.026388	1 257 745	-	-	-
		2	1.035471	178 318	-	-	-
Laspeyres	1.028025	1	1.026761	1 243 742	-	-	-
		2	1.036621	182 775	-	-	-
Paasche	1.026968	1	1.025979	1 306 914	-	-	-
		2	1.034194	180 275	-	-	-
Marshall-Edgeworth	1.027492	1	1.026365	2 517 564	-	-	-
		2	1.035436	357 090	-	-	-
Walsh-2	1.027465	1	1.026329	1 273 094	-	-	-
		2	1.035476	181 355	-	-	-
Walsh-Vartia	1.027425	1	1.026291	1 243 742	1 306 914	-	-
		2	1.035443	182 775	180 275	-	-
Theil	1.027563	1	1.026441	1 273 834	-	-	-
		2	1.035475	181 411	-	-	-
Vartia	1.027564	1	1.026442	1 243 742	1 306 914	-	-
		2	1.035475	182 775	180 275	-	-
Banerjee (GUV-3)**	1.027503	1	1.026375	1 243 742	1 306 914	2 580 736	-
		2	1.035428	182 775	180 275	354 590	-
Davies (GUV-4)**	1.027547	1	1.026419	1 243 742	1 306 914	1 289 079	-
		2	1.035449	182 775	180 275	177 170	-
(GUV-5)**	1.027589	1	1.026463	1 243 742	1 306 914	643 902	-
		2	1.035469	182 775	180 275	88 523	-
(GUV-6)**	1.027457	1	1.026333	1 243 742	1 306 914	1 291 609	-
		2	1.035437	182 775	180 275	177 237	-
Lehr (GUV-7)**	1.027500	1	1.026376	1 243 742	1 306 914	1 290 319	-
		2	1.035458	182 775	180 275	177 113	-

∗ Source: Statistics Sweden, Consumer Price Index Data for 2010-2011

∗∗ See Auer (2014, pp. 850-52).

10. Some Additional Requirements and Related Literature

The studies of Vartia (1976a, b) are the first formal treatments of consistency in aggregation. A more general definition of consistency in aggregation is proposed by Blackorby and Primont (1980, p. 96) who also introduce the notion of primary and secondary attributes. We take their definition as a starting point for the following discussion.

As in our Definition 6, Blackorby and Primont (1980, p. 96) postulate that a secondary attribute of some item i must exclusively use information that specifically relates to this item. In contrast to Definition 6, however, they allow only for quasi-additive aggregation of secondary attributes and they do not preclude "external information", that is, information other than prices and quantities (e.g., quality of an item). Blackorby and Primont (1980, p. 96) are well aware of the (too) general nature of their approach. They conclude: "Thus, unless there is some *a priori* notion of how the attributes are defined, this generalized consistency-in-aggregation notion does not seem helpful."

The present paper has argued that in the specific context of price measurement such an *a priori* notion exists. In a price index computation, the only available pieces of information are those in the informational set I. The secondary attributes must exclusively use information from set I, that is prices and quantities, or equivalently, price ratios, r_i, and expenditures, v_i^0 and v_i^1. Accordingly, Definition 6 precludes any "external information".

Nevertheless, some practitioners may still regard Definition 6 as too general, because it neglects some additional requirements that one possibly wants to attach to the secondary attributes and their aggregation. We discuss four increasingly restrictive requirements (Requirements A to D).

In Sections 7 and 8 we listed fifteen price indices that are consistent in aggregation with respect to some secondary attribute $\mathbf{z}$ in the sense of Definition 6. Adding the new requirements reduces the number of price indices that are considered as consistent in aggregation. Only four out of the fifteen price indices satisfy all four requirements. Unfortunately, there is no agreement as to which requirements are sensible and necessary and which are not. So far, the different positions are not thoroughly related to each other and it would be overoptimistic to expect a complete agreement on the issue. However, our formalized exposition adds more structure to the dispute and, as a result, may create a greater consensus. Definition 6 in conjunction with the list of additional requirements enables us to pinpoint the differences in past attempts of defining consistency in aggregation. Therefore, the following discussion also provides a comprehensive review of the price index literature on consistency in aggregation.

From an economic perspective, aggregating the quantities of different items (e.g., cornflakes and cars) is a meaningless operation. However, it makes sense to aggregate the *monetary values* of such items. Therefore, the secondary attributes should be measured in monetary units. The following requirement formalizes this postulate.

Requirement A. All secondary attributes are linearly homogeneous with respect to the expenditures:

$$z^q(r_i, \lambda v_i^0, \lambda v_i^1) = \lambda z^q(r_i, v_i^0, v_i^1) \ ,$$

for $q = 1, \ldots, Q$.

For example, the secondary attributes v_i^0, $\sqrt{v_i^0 v_i^1}$, or $\sqrt{v_i^0 v_i^1 / r_i}$ represent monetary units, whereas r_i, $\sqrt{v_i^0}$, and $v_i^0 v_i^1$ do not. All price indices listed in Tables 2 and 3 fulfill Requirement A. This is true also for the superlative indices of Fisher and Walsh. However, the Törnqvist index utilizes r_i as a secondary attribute. Therefore, it violates Requirement A.

A price index that is consistent in aggregation with respect to a secondary attribute $\mathbf{z}$, is a consistent aggregation rule, A_n, that determines how the individual values of each secondary attribute, z_i^q ($q = 1, \ldots, Q$ and $i = 1, \ldots, n$), are transformed into the respective aggregated value, Z^q. It seems reasonable to postulate that in this transformation an aggregated value, Z^q, depends only on the individual values of the secondary attribute q: $z_1^q, \ldots, z_n^q$. Which types of transformation are acceptable? Since the secondary attributes, z_i^q, must be expressed in monetary units (Requirement A), one may demand that also the aggregated values of the secondary attributes, Z_k^q, must be expressed in monetary units. Simple summation of the individual z_i^q-values preserves the units of measurement. Therefore, the preceeding demands can be combined in the following requirement:

Requirement B. The secondary attributes are aggregated additively:

$$A_n\left((r_1, z_1^1, \ldots, z_1^Q), \ldots, (r_n, z_n^1, \ldots, z_n^Q)\right) = \left(P_n, \sum_{i=1}^n z_i^1, \ldots, \sum_{i=1}^n z_i^Q\right) .$$

This precludes other quasi-additive aggregator functions such as multiplication. Again, all listed price indices, except for the Törnqvist index, satisfy Requirement B.

Consensus condition (ii) stated that on both stages of a two stage computation the "same functional form" must be applied. As an extensive interpretation of this condition one may postulate that any functional relationship between the secondary attributes of the individual items must carry over to the aggregated secondary attributes. For example, consider the four secondary attributes of the Fisher index (14). They are linked by

$$z_i^3 = r_i z_i^1 \qquad \text{and} \qquad z_i^4 = z_i^2 / r_i \ .$$

Let $Z^1 = \sum_{i \in S} z_i^1$, $Z^2 = \sum_{i \in S} z_i^2$, $Z^3 = \sum_{i \in S} z_i^3$, and $Z^4 = \sum_{i \in S} z_k^4$ denote the aggregate values of the secondary attributes. Since

$$Z^3 \neq P Z^1 \qquad \text{and} \qquad Z^4 \neq Z^1 / P \ ,$$

the relationships between the secondary attributes of the Fisher index (14) do not carry over to their aggregated counterparts. More formally, the postulate can be stated in the following way:

Requirement C. If a map g exists, such that $z_i^q = g(r_i, \mathbf{z}_i^{-q})$, where $\mathbf{z}_i^{-q}$ is the vector of all secondary attributes except for attribute q, then $Z^q = g(P, \mathbf{Z}^{-q})$, where Z^q is the aggregate value of all z_i^q with $i \in S$, P is the price index with respect to set S, and $\mathbf{Z}^{-q}$ are the aggregated values of all secondary attributes except for attribute q.

The Walsh index (11) has only one secondary attribute: $z_i = \sqrt{v_i^0 v_i^1 / r_i}$. Therefore, no violation of Requirement C can arise. A price index with two secondary attributes may or may not satisfy Requirement C. For example, the Walsh-Vartia index,

$$\ln P^{\mathrm{WV}} = \sum_{i \in S} \frac{\sqrt{v_i^0}}{\sqrt{\sum_{j \in S} v_j^0}} \frac{\sqrt{v_i^1}}{\sqrt{\sum_{j \in S} v_j^1}} \ln r_i \ ,$$

can be interpreted as a price index with the two secondary attributes v_i^0 and v_i^1. Between these two attributes no functional relationship exists. Accordingly, Requirement C is fulfilled. In contrast, when a price index has more than two secondary attributes (e.g., all price indices listed in Table 3 and the Fisher index), Requirement C is usually violated. The Walsh index and all price indices listed in Table 2 fulfill Requirements A, B, and C.

Some former studies on consistency of aggregation added to Requirements A, B, and C an even more restrictive requirement (van Yzeren, 1958, p. 432; Vartia 1976a, p. 89; 1976b, pp. 124-125; Stuvel, 1989, p. 36; Balk, 1995, p. 85; 1996, p. 360; 2008, p. 109; Pursiainen, 2005, p. 21; 2008, p. 18):

Requirement D. Only the secondary attributes v_i^0 and v_i^1 are admissable.

Since the Walsh, Marshall-Edgeworth, Walsh-2, and Theil indices violate Requirement D, the only remaining price indices are the Laspeyres, Paasche, Walsh-Vartia, and Vartia indices.

Is it possible to provide some justification for Requirement D? In an extremely extensive interpretation of consensus condition (ii) one may postulate that the relationship between a secondary attribute z_i^q and the three basic variables r_i, v_i^0, and v_i^1 from which this attribute is computed, must carry over to the aggregated values. For example, the Marshall-Edgeworth index, P^{ME}, has the secondary attribute $z_i = v_i^0 + v_i^1/r_i$. However, for the aggregate value $Z = \sum_{i \in S} z_i$ we get $Z \neq V^0 + V^1/P^{\mathrm{ME}}$, with $V^t = \sum_{i \in S} v_i^t$. Therefore, the relationship for the individual items does not carry over to their aggregated counterparts. In fact, formal correspondence between the computation of a z_i^q-value and the computation of its aggregated counterpart Z^q requires that $z_i^1 = v_i^0$ and/or $z_i^2 = v_i^1$, that is, Requirement D.

Auer (2004, pp. 386-391) criticises Requirement D as being too restrictive and proposes a milder version. Besides v_i^0 and v_i^1 he allows also for the "hybrid" secondary attributes $v_i^0 r_i = p_i^1 x_i^0$ and $v_i^1/r_i = p_i^0 x_i^1$. With these four admissable secondary attributes, the Marshall-Edgeworth index would re-enter the list of price indices that are consistent in aggregation.

Definition 6 does not award the label "consistent in aggregation". Instead, it awards the label "consistent in aggregation with respect to a secondary attribute **z**". We propose to reserve the label "consistent in aggregation" for those price indices that satisfy Definition 6 and from Requirements A to D those that are deemed as indispensable.

Unfortunately, there is no consensus on the list of indispensable requirements. In Table 4 we demonstrated for each of the fifteen price indices that they compute the overall inflation in a consistent two stage procedure. Therefore, some index users may reject all four requirements. Then, all price indices listed in Table 4 are consistent in aggregation.

Some index users may consider Requirements A to B as indispensable, but not the other two requirements. This would delete the Törnqvist index from the list of price indices that are consistent in aggregation. If Requirements A to C are deemed as compulsory, the Fisher index and all price indices listed in Table 3 drop out. If index users regard all four requirements as indispensable, then the Walsh, the Marshall-Edgeworth, the Walsh-2, and the Theil indices would no longer qualify for the label "consistent in aggregation".

The latter four indices share with the Laspeyres and the Paasche indices another property that is often appreciated in applied work. The indices can be written in the following form:

$$P - 1 = \sum_{k=1}^{K} (P_k - 1) \frac{Z_k}{\sum_{l=1}^{K} Z_l} , \qquad (19)$$

with $Z_k = \sum_{i \in S_k} z_i$. For example, in Sections 5 and 9 we applied the Walsh index to the Swedish consumer price index data and obtained $P^{\mathrm{Wa}} = 1.0275$, that is, an overall inflation of 2.75%. In applied work one may want to decompose these 2.75% into the contribution of the core inflation and the contribution of the non-core inflation. We know from our calculations that the core inflation was 2.64% ($P_1^{\text{Wa}} = 1.0264$) and the non-core inflation 3.55% ($P_2^{\text{Wa}} = 1.0355$). To compute the individual contributions, though, these two numbers are not sufficient. We also need weights that reflect the importance of the items assigned to core inflation relative to the items assigned to non-core inflation. These weights can be obtained from the secondary attribute. In Equation (19) the weight of each subset k is quantified by $Z_k \Big/ \sum_{l=1}^{K} Z_l$. In our Swedish example, Equation (19) becomes

$$P^{\text{Wa}} - 1 = \left(P_1^{\text{Wa}} - 1\right) \frac{Z_1}{Z_1 + Z_2} + \left(P_2^{\text{Wa}} - 1\right) \frac{Z_2}{Z_1 + Z_2} \; . \tag{20}$$

The aggregated values of the secondary attributes were $Z_1 = 1\,257\,744.8$ and $Z_2 = 178\,317.6$. Inserting all numbers in (20) yields

$$2.75 = 2.31 + 0.44 \; .$$

Even though the core inflation is much smaller than the non-core inflation, the contribution of the core inflation to the overall inflation of 2.75% is 2.31%, whereas the contribution of the non-core inflation is merely 0.44%. A detailed exposition of the decomposition properties of price indices can be found in Balk (2008, pp. 140-151).

11. Concluding Remarks

The computation of the overall price change is often conducted in a two stage (or multi stage) procedure, where on the first stage price changes of subgroups are computed and on the second stage the price changes of the subgroups are aggregated into the overall price change. In the literature it has been postulated that the price index formula applied in such a multi stage analysis should be consistent in aggregation. The present paper has argued that consistency in aggregation is a complex concept that requires a careful definition. Following Blackorby and Primont (1980), the definition distinguishes between the primary attribute of a price index and its secondary attributes. Combining this distinction with the general concept of a consistent aggregration rule, yields a thoroughly motivated basic definition of consistency in aggregation, specifically designed for the context of price measurement.

Surprisingly many price index formulae satisfy this basic definition of consistency in aggregation. Among these are the superlative price indices of Fisher, Törnqvist, and Walsh. This is a remarkable finding, because these indices are known as superlative price indices. In the literature there has been a general perception that superlative price indices are particularly reliable for single stage computations, but that they are not consistent in aggregation and therefore unsuitable for multi stage computations. Our findings show that this perception cannot be sustained in the context of our new definition of consistency in aggregation.

It was argued that further requirements can be added to the basic definition of consistency in aggregation. Such additional requirements shrink the list of price indices that are consistent in aggregation. Four such requirements were discussed, with Requirements A and B being the most obvious ones, and Requirement D being the most contentious one. From an applied perspective, Requirements A to C appear particularly relevant. The Törnqvist index satisfies none of them, the Fisher index satisfies Requirements A and B, whereas the Walsh index satisfies all three requirements. Therefore, empirical researchers that regard superlative price indices as particularly reliable and want to compile some overall price change by a multi stage procedure, may apply the Walsh index for this purpose.

Acknowledgements

We are grateful to Marcel Greuel for his extremely valuable research assistance. We also have had the benefit of useful suggestions from Bjørn Kjos-Hanssen and Leonhard Frerick. The paper was presented at the 15th Meeting of the Ottawa Group (May 2017, Eltville) where we received helpful comments by Bert Balk, Jörgen Dalén, Erwin Diewert, Jens Mehrhoff, Tsutomu Watanabe, and Li-Chun Zhang. This research did not receive any specific grant from funding agencies in the public, commercial, or not-for-profit sectors.

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Appendix

Proposition 7. *Without the continuity requirement in Definition 6, every symmetric price index would be consistent in aggregation with respect to some secondary attribute.*

Proof. Considered as a $\mathbb{Q}$-vector space the reals are $\mathbb{R}$-dimensional and using the axiom of choice as well as the fact that $J = \mathbb{R}^3_{>0}$ and $\mathbb{R}$ have the same cardinality we can thus take a Hamel basis $\{e_a : a \in J\}$ of $\mathbb{R}$. Let M be the set of all finite linear combinations of elements e_a with (strictly positive) integer coefficients and define $z : I \to M$ by $z(a) = e_a$.

The linear independence then implies for all $a_1, \ldots, a_n, b_1, \ldots, b_m \in J$ that

$$\sum_{i=1}^{n} z(a_i) = \sum_{i=1}^{m} z(b_i) \;\Rightarrow\; n = m \text{ and } a_i = b_{\pi(i)} \text{ for some permutation } \pi.$$

In order to apply Proposition 3 (more precisely, the version neglecting the continuity aspects), we want to define a function $f : \mathbb{R}_{>0} \times M \to \mathbb{R}_{>0}$ such that

$$f\left(P_n(a_1, \ldots, a_n), \sum_{i=1}^{n} z_i\right) = \sum_{i=1}^{n} f(r_i, z_i)$$

for all $n \in \mathbb{N}$, $a_i = (r_i, v_i^0, v_i^1) \in J$, and $z_i = z(a_i)$ so that all partial functions $r \mapsto f(r, m)$ are invertible.

Given $m \in M$ there are (up to the order) unique $a_1, \ldots, a_n \in J$ with $m = z(a_1) + \cdots + z(a_n)$. We then set $\alpha(m) = P_n(a_1, \ldots, a_n)$, $\beta(m) = r_1 + \cdots + r_n$ (where, as previously, r_i is the first component of a_i), and

$$f(r, m) = r\beta(m)/\alpha(m).$$

Of course, the partial functions $r \mapsto f(r, m)$ are invertible on $\mathbb{R}_{>0}$.

In order to show the condition of Proposition 3 we take $n \in \mathbb{N}$ and $a_i = (r_i, v_i^0, v_i^1) \in J$. For $z_i = z(a_i)$ we have $\alpha(z_i) = P_1(a_1) = r_1$ and $\beta(z_i) = r_i$ so that

$$f(r_i, z_i) = r_i r_i / r_i = r_i.$$

Moreover, for $m = z_1 + \cdots + z_n$ and $\varrho = P_n(a_1, \ldots, a_n)$ we have $\alpha(m) = \varrho$ and $\beta(m) = r_1 + \cdots + r_n$ and hence

$$f\left(P_n(a_1, \ldots, a_n), \sum_{i=1}^{n} z_i\right) = \varrho\beta(m)/\alpha(m) = \sum_{i=1}^{n} r_i = \sum_{i=1}^{n} f((r_i, z_i))$$

which completes the proof. ■