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A macroeconomic reverse stress test

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Non-technical summary

Research Question

In response to the financial crisis 2007-2009, regulatory authorities have strengthened the importance of stress test methodologies and particularly emphasized the role of reverse stress tests. Reverse stress tests look exactly for those scenarios which lead to a very unfavorable event for a bank, for example, an equity-exhausting loss, a non-fulfillment of the capital adequacy requirements or illiquidity. More generally, scenarios shall be identified that lead to an outcome in which the bank's business plan becomes unviable and the bank insolvent. In this paper, we show how a fully-fledged macroeconomic reverse stress test for credit and interest rate risk can be implemented which is in line with the new regulatory requirements.

Contribution

This paper contributes to the sparse quantitative reverse stress test literature and sketches a framework which allows to model interactions between different risk factors at the level of individual financial instruments and risk factors. The focus lies on the presentation of the calibration procedure and on a detailed discussion of practical implementation issues.

Results

We search for the most likely scenario which exhausts the bank's equity. It turns out that this so-called reverse stress test scenario, which is given by a combination of macroeconomic variables, is economically reasonable for the assumed bank portfolio. In particular, for a bank which engages in maturity-transformation, the most likely reverse stress test scenario implies a steeper interest rate curve and an economic downturn. However, the paper also reveals that due to high data requirements and intensive computational efforts, reverse stress tests are exposed to considerable model and estimation risk which makes numerous robustness checks necessary.

Nichttechnische Zusammenfassung

Fragestellung

Als Reaktion auf die Finanzmarktkrise 2007-2009 hat die Bankenaufsicht die Bedeutung von Stresstests herausgestellt und dabei insbesondere die wichtige Rolle von inversen Stresstests betont. Im Rahmen von inversen Stresstests sollen Szenarien identifiziert werden, die dazu führen, dass das Geschäftsmodell einer Bank nicht mehr tragfähig ist und die Bank die Grenze zur Insolvenz überschreitet (z.B. aufgrund hoher Verluste, der Nichteinhaltung von regulatorischen Kapitalanforderungen oder Illiquidität). Dieses Papier stellt einen makroökonomischen inversen Stresstest für Kredit- und Zinsrisiken vor, der die neuen regulatorischen Vorgaben erfüllt.

Beitrag

Das Papier erweitert den Literaturstrang zu quantitativen inversen Stresstests, und zeigt ein Modell auf, welches Interaktionen zwischen unterschiedlichen Risikoarten auf der Ebene von einzelnen Finanzinstrumenten und Risikofaktoren berücksichtigt. Der Schwerpunkt liegt auf der Darstellung des Kalibrierungsprozesses und auf einer ausführlichen Diskussion von praktischen Umsetzungsproblemen.

Ergebnisse

Gesucht wird das wahrscheinlichste Szenario, welches das Eigenkapital einer Bank aufbraucht. Es zeigt sich, dass dieses aus einer Kombination von makroökonomischen Variablen bestehende inverse Stresstest-Szenario für das angenommene Portfolio ökonomisch sinnvoll ist und daher der vorgeschlagene inverse Stresstest zu sinnvollen Ergebnissen führt. Für eine Bank, die positive Fristentransformation betreibt, impliziert das wahrscheinlichste inverse Stresstest-Szenario eine steilere Zinsstrukturkurve und einen ökonomischen Abschwung. Allerdings wird auch deutlich, dass aufgrund hoher Datenanforderungen und eines großen Berechnungsaufwands, quantitative inverse Stresstests erheblichen Modell- und Schätzrisiken ausgesetzt sind, was zahlreiche Robustheitsüberprüfungen notwendig macht.

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Abstract

Reverse stress tests are a relatively new stress test instrument that aims at finding exactly those scenarios that cause a bank to cross the frontier between survival and default. Afterward, the scenario which is most probable has to be identified. This paper sketches a framework for a quantitative reverse stress test for maturity-transforming banks that are exposed to credit and interest rate risk and demonstrates how the model can be calibrated empirically. The main features of the proposed framework are: 1) The necessary steps of a reverse stress test (solving an inversion problem and computing the scenario probabilities) can be performed within one model, 2) Scenarios are characterized by realizations of macroeconomic risk factors, 3) Principal component analysis helps to reduce the dimensionality of the space of systematic risk factors, 4) Due to data limitations, the results of reverse stress tests are exposed to considerable model and estimation risk, which makes numerous robustness checks necessary.

Keywords: copula functions, extreme value theory, principal component analysis, reverse stress testing

JEL classification: C22, C51, C53, G21, G32

*Corresponding author: Kamil Pliszka, Deutsche Bundesbank, Wilhelm-Epstein-Str. 14, 60431 Frankfurt am Main. Phone: +49-69-9566-6815. E-mail: kamil.pliszka@bundesbank.de. Previously, the paper circulated under the title 'Empirical implementation of a quantitative reverse stress test for defaultable fixed-income instruments with macroeconomic factors and principal components'. We wish to thank the participants of the Bundesbank Seminar, the FEBS conference (Paris, 2013), the FMA - European Conference (Luxembourg, 2013), the EFMA conference (Reading, 2013) and the CREDIT conference (Venice, 2013) for their helpful comments. This paper represents the authors' personal opinions and does not necessarily reflect the views of the Deutsche Bundesbank or its staff.

1 Introduction

In response to the financial crisis 2007-2009, regulatory authorities have strengthened the importance of stress test methodologies. In particular, the role of reverse stress tests was emphasized following a number of consultative papers by the Financial Services Authority (FSA (2008, 2009)) and the Committee of European Banking Supervisors (CEBS (2009, 2010)). Large banks are expected to perform reverse stress tests in a quantitative way. However, up to now, no appropriate standard for this kind of stress test has evolved and even the number of (at least) published proposals on how such a test might be performed at all is very limited.

In regular stress tests, adverse scenarios are chosen upon historical observations or expert knowledge. Thus, although the choice may be reasonable, the employed scenarios remain arbitrary. In contrast, in reverse stress tests, exactly those scenarios are looked for that lead to a very unfavourable event for a bank (e.g., a very large (expected) loss, a non-fulfillment of the capital adequacy requirements or illiquidity). More generally, scenarios shall be identified that lead to an outcome in which the bank's business plan becomes unviable and the bank insolvent. In the next step, the most plausible of these scenarios has to be found and evaluated by the bank's senior management (see CEBS (2010, p. 20)). Čihák (2007) calls this the "threshold approach". Reverse stress testing is mathematically and conceptually challenging, particularly, if many risk factors are relevant to the value of the bank's portfolio and when this portfolio is structured in a complex way with many different assets and financial instruments. For n risk factors, n-dimensional scenarios have to be found when solving the inversion problem inherent in a reverse stress test and, for each single scenario, the corresponding probability of occurrence has to be computed. Therefore, the number of considered risk factors has to be kept low and a framework has to be chosen that remains numerically tractable for more sophisticated portfolios.

Most of the literature dealing with macroeconomic regular stress tests for credit risk is based on the idea of Wilson (1997a, 1997b) and extensions thereof. Within this type of model, macroeconomic variables are looked for that can explain the systematic variation of default rates across time (see, for example, Boss (2002), Sorge and Virolainen (2006)). The current body of literature on reverse stress tests is still sparse. A discussion of a qualitative approach based on fault trees has been presented by Grundke (2012b). However, the essential conclusion of this paper is that a qualitative approach alone would not work, or, at least, would have to be supported by quantitative elements. Füser, Hein, and Somma (2012a, 2012b) present a very general operating plan for (mainly qualitative) reverse stress tests. Papers on quantitative reverse stress tests are also very rare. One approach is developed by Grundke (2011). Employing ideas from integrated risk measurement, he calculates a bank's loss in economic value of equity using a CreditMetrics-based bottom-up model with correlated interest rates and rating-specific credit spreads. Later, in Grundke (2012a), this approach is expanded by more realistic assumptions, including, among other things, contagion effects between single obligors and a time-varying bank rating. Drüen and Florin (2010) argue in a vein similar to Grundke (2011), but they do not use a fully fledged bottom-up approach. Instead, they prefer to employ two separate approaches for interest rate risk and default risk and, additionally, some exogenous (not further described) functional relationship between shifts in the term structure of risk-free interest rates and the obligors' default probabilities. Furthermore, there are some case studies for simply structured portfolios with one or two risk factors (see, for example, Liermann and Klauck (2009)). Beside this, a dimension reduction technique that yields the most relevant (based on information criteria) risk factors of a portfolio has been proposed by Skoglund and Chen (2009). A more recent paper by McNeil and Smith (2012) introduces the concept of depth to identify the most plausible reverse stress test scenario, which is called the "most likely ruin event" (MLRE). In a related strand of stress test literature, the worst (in the sense of 'expected losses for a given portfolio') scenario from a set of scenarios with a given plausibility (for example, measured by the Mahalanobis-distance) is looked for. Čihák (2007) calls this the "worst case approach". Examples of this approach are Breuer, Jandačka, Rheinberger, and Summer (2008), Breuer, Jandačka, Rheinberger, and Summer (2010) and Breuer, Jandačka, Mencia, and Summer (2012).

Our approach picks up ideas from the framework of Grundke (2011, 2012a). However, instead of performing pure simulation studies, we show how a quantitative reverse stress test can be implemented empirically using U.S. data. Furthermore, we use principal components for reducing the number of rate-sensitive risk factors relevant to defaultable fixed-income instruments (see, for example, Jamshidian and Zhu (1997) in a pure interest rate risk setting). This specification keeps the dimensionality of the model low and, hence, allows us to specify and to calibrate a full reverse stress test framework. Finally, the issues of model and estimation risk are considered. The single methodological components of our reverse stress test are already known from the interest rate and credit portfolio modeling literature. Our contribution is to show how these components can be combined and implemented for fulfilling the new regulatory requirements regarding reverse stress tests. The general framework that we present is applied by way of example to a stylized maturity-transforming bank. However, although the bank's portfolio is composed of simple defaultable fixed-income instruments, the framework is general enough to be extendable to cover other risk factors, too (e.g., currency risk and equity risk).

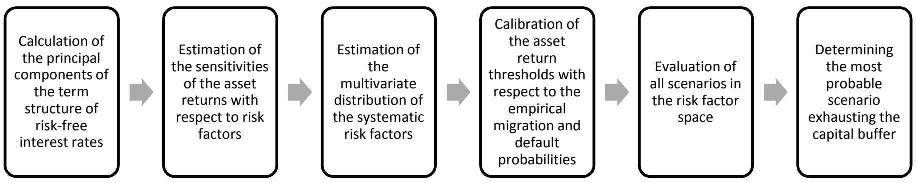

Figure 1: Outline of the reverse stress test. First, the principal component analysis for the term structure of risk-free interest rates is carried out and the linear factor model describing the asset returns of the bank's obligors is estimated (steps 1 and 2). Next, the univariate margins of the risk factors and the multivariate dependence structure are analyzed (see steps 3 and 4). Then, the results of a reverse stress test for a stylized fixed-income portfolio with credit risk are presented (steps 5 and 6).

The remainder of the paper is structured as follows. In Section 2, the methodology of the paper is presented. Section 3 presents the data. In Section 4, the results of the model calibration and of the reserve stress test are shown. First, the principal component analysis for the term structure of risk-free interest rates is carried out and the linear factor model for asset returns of the bank's obligors is estimated by maximum likelihood (steps 1 and 2 in Figure 1). Next, the univariate margins of the risk factors and the multivariate dependence structure are analyzed (see steps 3 and 4). Then, it is demothe results of a reverse stress test for a stylized fixed-income portfolio with credit risk are presented (steps 5 and 6). In Section 5, we discuss our reverse stress test procedure with respect to issues of practical implementation. Finally, Section 6 concludes.

2 Methodology[1]

2.1 Reverse stress test

In this section, we describe how the actual reverse stress test works and what the stylized bank portfolio to which the modeling framework is applied is composed of. Like Grundke (2011, 2012a), we assume a bank portfolio that exclusively consists of assets and liabilities structured as zero-coupon bonds. The bank pursues a strategy of positive maturity transformation implying negative net cash flows in the short term and positive net cash flows in the long term. In particular, we have a huge negative net cash flow in the shortest time bucket and, then, increasing net cash flows for the following maturities, passing over to constant ones as they finally decrease for the last two maturities. Moreover, it is assumed that the term structure of the bank's assets and liabilities does not vary across time. Thus, value variations caused by a decreasing time to maturity are not considered. We assume a cash flow profile as illustrated in Figure 2 to be representative for maturity-transforming banks.

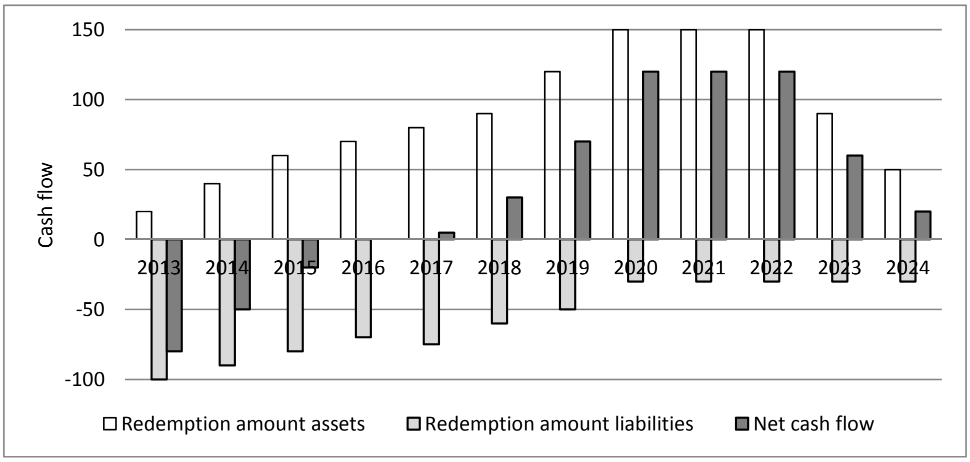

Figure 2: Cash flows of assets and liabilities of the stylized bank. The bank pursues a strategy of positive maturity transformation, where it is assumed that the term structure of the bank's assets and liabilities does not vary across time.

All defaultable zero-coupon bonds $n \in \{1, ..., N\}$ on the asset side are assumed to be issued by different obligors with initially equal default probability. They have a standardized redemption amount of one and a time to maturity of $T_n \in \{1, ..., 12\}$.

In general, our proposed framework is flexible enough to consider more complex financial instruments, however, at the cost of larger estimation efforts and a higher computational burden. The situation would get even more intricate when, additionally, we consider instruments that, conditional on the realization of the risk factors, have no fixed cash flows,

[1] A symbol directory for Section 2.1 to Section 2.3 is given in the Appendix A in Table 10 to Table 12.

but for which behavioral assumptions are needed to determine the involved cash flows, for example, withdrawals of saving accounts, drawings of credit lines or cancellations of fixed-rate loans with prepayment option. Although there are several commonalities among banks, in general, the specification of these behavioral assumptions and the extent of embedded optionalities highly depend on a bank's business model and on its jurisdiction (see BIS (2006a, p. 212)). These behavioral assumptions would add further model risk to the reverse stress test framework. Finally, despite the simplifications that we assume with respect to our stylized bank portfolio, it has to be stressed that in most real-world bank portfolios, credit and interest rate risk (i.e., the risks that we model) are the most important risk types. Furthermore, the assumed cash flow profile (see Figure 2) corresponds to positive maturity transformation which is a strategy that can be observed in many banks. For the results of the reverse stress test, it is irrelevant whether this cash flow profile is induced by zero-coupon bonds, coupon bonds or more complex financial instruments.

The value of a defaultable zero-coupon bond at the risk horizon H issued by obligor n who is rated as $\zeta_H^n \in \{1,2,3,4,5,6,7\} = \{\text{AAA, AA, A, BBB, BB, B, C-CCC}\}$ at the risk horizon H is given by

$$B^d(C_1(H),...,C_p(H),\zeta_H^n,T_n) = \exp\left\{-(R(C_1(H),...,C_p(H),T_n) + S_{\zeta_H^n}) \cdot T_n\right\} \tag{2.1}$$

where $R(C_1(H),...,C_p(H),T_n)$ denotes the stochastic risk-free interest rate at the risk horizon H for a time to maturity of T_n, which is calculated from the last observed risk-free interest rate at time $t = 0$ and the first p principal components of the risk-free interest rate curve at time $t = H$ by

$$R(C_1(H),...,C_p(H),H,T_n) = r_{T_n}(0) \cdot (1 + \Delta r_{T_n}(H)) = r_{T_n}(0) \cdot \Big(1 + \sum_{j=1}^{p} c_{T_n,j} \cdot C_j(H)\Big). \tag{2.2}$$

The expression $S_{\zeta_H^n}$ denotes the average (over times to maturity and obligors with the same rating grade) non-stochastic credit spread for rating grade ζ_H^n at the risk horizon H.[2] To model the recovery payment to the bank in the event of a default by an obligor, we apply a modified recovery-of-treasury assumption.[3] In the case of a default by obligor n, the minimum of a beta-distributed fraction δ_n with the empirically observed parameters $\mu_{B^d} = 0.518$ and $\sigma_{B^d} = 0.389$ of a risk-free, but otherwise identical, zero-coupon bond, and the value of the bond without any rating transition of the obligor between 0 and H, is paid.[4] This convention ensures that the payment in the event of a default is never

[2]Obviously, the pricing approach could be easily modified to consider non-stochastic credit spreads that depend on the time to maturity. More data-intensive would be an approach in which the credit spreads of different times to maturity and rating grades are modeled by a multivariate distribution (under full consideration of existing dependencies).

[3]See Grundke (2011, 2012a).

[4]The mean and the standard deviation of the beta-distributed recovery rate equal Standard & Poor's mean and standard deviation of the recovery rate of senior unsecured bonds during 1987 to 2011 (see Standard & Poor's (2011b)).

larger than the previous value of the bond.[5] As in the original CreditMetrics model, the recovery rates are assumed to be independent across issuers and independent of all other stochastic variables in the model.

The values of the positions $v \in \{1, ..., V\}$ on the liability side are given by

$$B^l(C_1(H), ..., C_p(H), T_v) = \exp\{-(R(C_1(H), ..., C_p(H), T_v) + S_{\zeta_H^{bank}=AA}) \cdot T_v\}. \quad (2.3)$$

This representation uses the assumption that the bank is initially rated as AA and is not exposed to migration risk until the risk horizon.[6]

To simplify calculations, we impose a homogeneity assumption with respect to the initial credit quality of the bank's asset portfolio: At time $t = 0$, the obligors on the asset side are assumed to be exclusively rated as AA ($\zeta_0^n = AA \; \forall \; n \in \{1, ..., N\}$) and BB ($\zeta_0^n = BB \; \forall \; n \in \{1, ..., N\}$), respectively.

The market value of the bank's equity at the risk horizon H is given by the difference between the sum of the market values of the assets and the sum of the market values of the liabilities at the risk horizon H:

$$V_E(H) = \sum_{n=1}^{N} B^d(C_1(H), ..., C_p(H), \zeta_H^n, T_n) - \sum_{v=1}^{V} B^l(C_1(H), ..., C_p(H), T_v). \quad (2.4)$$

As in Grundke (2011, 2012a), a scenario $\boldsymbol{\omega}$ is composed of realizations of systematic risk factors. It is classified as a reverse stress test scenario when the existing capital buffer B (defined as the initial market value $V_E(0)$ of the bank's equity) is consumed by a conditional decrease in the expected equity value at the risk horizon H and by the respective conditional economic capital requirement. Thus, a bank's default is understood as a non-fulfillment of the economic capital requirements according to the second pillar of Basel II. When the value-at-risk at a confidence level of α is used as an economic capital measure and is defined as the difference between the conditional expected equity value at the risk horizon H and the $(1-\alpha)$-quantile of the conditional probability distribution of

[5] This assumption proves to be very sensible. Due to the high volatility of the recovery rate, we can observe that the pure recovery-of-treasury assumption would lead in a surprisingly large number of cases to higher values of bonds after default. A simulation within our framework reveals that, for AA-rated obligors, the fraction of increases in value after default ranges between 13.02% (default of obligors with a maturity of $t = 1$) and 28.98% (default of obligors with a maturity of $t = 12$), whereas in the case of BB-rated obligors, the fraction ranges between 22.45% (default of obligors with a maturity of $t = 1$) and 48.72% (default of obligors with a maturity of $t = 12$). Our modified version avoids this unfavorable effect.

[6] This assumption corresponds to an accounting standard under which firms are not allowed to consider value variations of their equity caused by changes in their own credit quality in a way that affects their net income. For an alternative modeling with time-varying bank rating, see Grundke (2012a). With a time-varying bank rating, care has to be taken to avoid circularity problems.

the bank's equity, the most likely reverse stress test scenario is given by

$$\arg\max_{\boldsymbol{\omega}\in\boldsymbol{\Omega}^*} P(\boldsymbol{\omega})$$
$$\text{with } \boldsymbol{\Omega}^* = \{\boldsymbol{\omega}\in\boldsymbol{\Omega} | \underbrace{\mathbb{E}[V_E(H)] - \mathbb{E}[V_E(H)|\boldsymbol{\omega}]}_{=\text{expected loss, if } \boldsymbol{\omega} \text{ occurs}} + \underbrace{\mathbb{E}[V_E(H)|\boldsymbol{\omega}] - q_{1-\alpha}(V_E(H)|\boldsymbol{\omega})}_{=VaR_{\alpha,H}(V_E(H)|\boldsymbol{\omega})} = B\}$$
$$= \{\boldsymbol{\omega}\in\boldsymbol{\Omega} | \mathbb{E}[V_E(H)] - q_{1-\alpha}(V_E(H)|\boldsymbol{\omega}) = B\}. \tag{2.5}$$

The above definition of the most likely reverse stress test scenario also makes clear how we resolve the trade-off between the plausibility of a stress scenario and the question of how extreme it is. Obviously, one can always find scenarios that cause larger losses at the price of a smaller probability of occurrence. The above definition of the set $\boldsymbol{\Omega}^*$ of reverse stress test scenarios shows that we only consider scenarios that are sufficiently extreme to consume the bank's existing capital buffer B. All other scenarios that are even more extreme are not of any interest. Afterward, the most plausible scenario (in the sense of the most likely scenario given the estimated multivariate distribution of the systematic risk factors) out of the set $\boldsymbol{\Omega}^*$ is identified.

For solving the optimization problem (2.5), a grid search in the space of the systematic risk factors is performed. For each grid point, we calculate the conditional value-at-risk of the bank's equity at the risk horizon H by Monte-Carlo simulation with $S = 1,000$ draws.[7] To evaluate all scenarios $\boldsymbol{\omega}$, for each systematic risk factor, a grid search is carried out within the interval $[\mu - 4\cdot\sigma, \mu + 4\cdot\sigma]$, which is split into equally-sized subintervals. We choose a step size of $0.5\cdot\sigma$, where σ is the standard deviation of a risk factor. Thus, we obtain 17 equidistant grid points per risk factor.

We assume that a latent systematic credit risk factor $Z(t)$, an observable economic indicator $X(t)$, principal components of the risk-free interest rate curve $C_1(t), ..., C_p(t)$ serve as systematic risk factors (see Section 2.2). Based on their multivariate distribution (see Section 2.3), the probability that a scenario $\boldsymbol{\omega} = (z, x, c_1, ..., c_p)$ occurs (defined as the probability that the realizations of the systematic risk factors lie within the bounds of the corresponding grid point) is computed as follows[8]

$$P(z^- < Z \le z^+, x^- < X \le x^+, c_1^- < C_1 \le c_1^+, ..., c_p^- < C_p \le c_p^+), \tag{2.6}$$

where the border points are given by

$$\begin{pmatrix} z^\pm & x^\pm & c_1^\pm & \dots & c_p^\pm \end{pmatrix} = \begin{pmatrix} z & x & c_1 & \dots & c_p \end{pmatrix} \pm 0.5 \cdot \text{factor-specific step size.} \tag{2.7}$$

[7]The idiosyncratic risk is the only source of uncertainty in the case of the conditional distribution. Therefore, the small number of Monte-Carlo simulation runs is sufficient.

[8]The expression for calculating probabilities on multi-dimensional intervals can be found, for example, in Mathar and Pfeifer (1990, p. 41). The computation of each probability term is done using the function pcopula of the package copula in the program R. In order to calculate probabilities, pcopula refers to the function pmvt of the package mvtnorm which uses randomized Quasi-Monte-Carlo methods (see, for example, Genz and Bretz (1999, 2002)). As the assigned probabilities on the edge of the considered part of the support are very low, numerical issues may lead to us obtaining implausible results, especially negative probabilities. To solve this problem, we calculate probabilities in the case of the t-copula as the mean over several repetitions.

2.2 Linear factor model and principal component analysis

Similar to Grundke (2011, 2012a), we assume that the credit quality of the bank's obligors is driven by their asset returns and that these asset returns are correlated with the risk-free interest rates.[9] The choice of risk factors is crucial in reverse stress tests and needs to go hand in hand with the considered portfolio. Various studies in the credit portfolio risk literature show that observable macro-financial risk factors and firm-specific risk factors are not enough to explain time-varying systematic credit risk (see Schwaab, Koopman, and Lucas (2014, p. 2), and the references cited therein). Thus, a latent systematic risk factor is introduced in the linear factor model for the asset returns to cover unobservable systematic credit risk (see similarly, for example, Rösch and Scheule (2007), or Breuer et al. (2012)). In sum, we assume that a latent systematic credit risk factor, an observable economic indicator, interest rate risk factors and an idiosyncratic risk factor influence each obligor's asset return. Formally, the complete linear factor model for the asset return $R_{n,i}(t)$ of obligor n, $n \in \{1, ..., N\}$, with rating grade $i, i \in \{1, ..., K\}$, within the time period $[t, t+1)$ is given by

$$R_{n,i}(t) = \sqrt{\rho_{i,Z}} \cdot Z(t) + \rho_{i,X} \cdot X(t) + \sum_{j=1}^{p} \rho_{i,C_j} \cdot C_j(t) + \sqrt{1-\rho_{i,Z}} \cdot \epsilon_n(t) \tag{2.8}$$

where $Z(t)$ is an i.i.d. standard normally distributed random variable representing latent systematic credit risk, $X(t)$ denotes an economic indicator, and $C_j(t)$, $j \in \{1, ..., p\}$, represent the principal components of the term structure of risk-free interest rates. The variable $\epsilon_n(t)$ denotes the idiosyncratic risk of obligor n at time t and is assumed to be an i.i.d. standard normally distributed random variable. The parameters $\sqrt{\rho_{i,Z}}$, $\rho_{i,X}$ and $\rho_{i,C_j}, j \in \{1, ..., p\}$, determine the sensitivity of the obligors' asset returns with respect to the systematic risk factors. If the bank's portfolio is not only composed of simple defaultable fixed-income instruments (as we assume, see Section 2.1), but includes, for example, also mortgage loans or options, further risk factors (such as house prices or volatility) would have to be considered for the pricing of the instruments at the risk horizon and possibly in the asset return equation (2.8). Indeed, the proposed framework is flexible enough to allow these extensions. But, of course, the estimation of the model and the solution of the inversion problem inherent in every reverse stress test would get more challenging.

In order to keep the number of risk factors low, we apply principal component analysis to explain the movements of the term structure of risk-free interest rates (see (2.1)). Principal component analysis reduces the dimensional complexity of a dataset by an orthogonal linear transformation of the original data into a new orthogonal space. Let r_q, $q \in \{1, 2, ..., m\}$, be the yield-to-maturity with time to maturity t_q. Then, the j-th

[9] The interpretation as asset returns results from the seminal Merton (1974) paper. More generally, the credit quality of an obligor is assumed to be driven by some creditworthiness index (see, for example, Dorfleitner, Fischer, and Geidosch (2012)). The lower the index is, the worse is the rating grade of the obligor. When the index is below a given threshold, this event is set equal to a default of the obligor.

principal component C_j is given by

$$C_j = \sum_{q=1}^{m} c_{j,q} \cdot \Delta r_q \tag{2.9}$$

where Δr_q denotes the percentage change of the q-th yield-to-maturity and $c_{j,q}$, $q \in \{1, 2, ..., m\}$, denotes the coefficients of the j-th principal component. Due to the assumed orthogonality of the coefficient matrix of the principal components, the yield-to-maturity changes Δr_q, $q \in \{1, 2, ..., m\}$, are given by linear combinations of the coefficients

$$\Delta r_q = \sum_{j=1}^{m} c_{q,j} \cdot C_j. \tag{2.10}$$

After determining the number of relevant principal components, which is represented by the variable p, the risk factor sensitivities in the asset return equation (2.8) have to be estimated. For this, we assume that the risk factor sensitivities vary for different initial rating grades $i, i \in \{1, ..., K\}$, of the obligors. The rating-specific log-likelihood function l_i takes a binomial shape as defaults are conditional on realizations of the systematic risk factors independent[10]

$$\begin{aligned} l_i = \sum_{t=1}^{T} \ln \int_{-\infty}^{+\infty} \binom{N_i(t)}{d_i(t)} q_i\big(z, x(t), c_1(t), ..., c_p(t)\big)^{d_i(t)} \\ \cdot \big(1 - q_i(z, x(t), c_1(t), ..., c_p(t))\big)^{N_i(t)-d_i(t)} \phi(z) dz, \end{aligned} \tag{2.11}$$

with the rating-specific conditional default probability[11]

$$\begin{aligned} & q_i\big(z, x(t), c_1(t), ..., c_p(t)\big) \\ := {} & P\Big(R_{n,i}(t) \leq R_{i,K} | Z(t) = z, X(t) = x(t), C_1(t) = c_1(t), ..., C_p(t) = c_p(t)\Big) \\ = {} & \Phi\left(\frac{R_{i,K} - \sqrt{\rho_{i,Z}} z - \rho_{i,X} x(t) - \sum_{j=1}^{p} \rho_{i,C_j} c_j(t)}{\sqrt{1 - \rho_{i,Z}}}\right). \end{aligned} \tag{2.12}$$

The above integral is solved using adaptive quadrature methods.[12] $\phi(z)$ ($\Phi(z)$) is the (cumulative) density function of a standard normally distributed random variable. $N_i(t)$ describes the number of obligors with rating grade i at time t and $d_i(t)$ is the number of defaults of obligors with rating grade i at time t within the period $[t, t+1)$. $R_{i,K}$ is the rating-specific default barrier, whose shortfall by an asset return is defined as a default of an obligor.

[10]Estimating factor loadings in linear factor models for asset returns by maximum likelihood (based on default data) is a frequently employed approach in the credit portfolio risk literature (see, for example, Gordy and Heitfield (2002), Frey and McNeil (2003), Hamerle and Rösch (2006), and Rösch and Scheule (2007)).

[11]An additional constraint $\rho_{i,Z} \in (0, 1)$ ensures that we do not divide by zero or compute the square root of a negative value.

[12]The implementation is done using the function int of the program R, which is based on the Gauss-Kronrod quadrature (see Kronrod (1965)).

2.3 Multivariate distributions

For computing the probabilities for reverse stress test scenarios, we need the multivariate probability distribution of the used systematic risk factors. These are the latent systematic credit risk factor $Z(t)$, the economic indicator $X(t)$, and the principal components $C_j(t)$, $j \in \{1, ..., p\}$, of the term structure of risk-free interest rates.

In order to compute the multivariate probability distribution, we estimate the marginal distributions of the risk factors and its multivariate relationship given by an unconditional copula function. Alternatively, for example, a multivariate time series model or univariate times series models with copula-dependent residuals could be estimated. However, due to the usually small number of data points, we refrained from doing this. For estimating the copula function, we do not have to take into account the latent systematic credit risk factor $Z(t)$ because this factor is assumed to be independent of all other variables (as usual in the literature on credit portfolio modeling).

As marginal distributions, we consider, for simplicity, the normal distribution and, when goodness-of-fit tests explicitly reject this distribution, a combination of the normal distribution (in the center) and the generalized Pareto distribution (GPD) that allows us to model heavier tails. The GPD quantifies the conditional distribution of excesses of a random variable X over a threshold u and is given by[13]

$$P(X - u \leq y | X > u) = G_{\xi,\beta}(y) = \begin{cases} 1 - \left(1 + \frac{\xi y}{\beta}\right)^{-\frac{1}{\xi}} & , \xi \neq 0 \\ 1 - \exp\left\{-\frac{y}{\beta}\right\} & , \xi = 0 \end{cases} \tag{2.13}$$

where $\beta > 0$ is referred to as the shape and ξ as the scale parameter. In the case of $\xi > 0$, fat tails are present.

Two popular types of copula functions are the elliptical and Archimedean copulas. Elliptical copulas, such as the normal copula and the t-copula, are derived from elliptical distributions. This type of copula is characterized by a symmetry of the dependence structure and, especially, (in case of the t-copula) by a symmetry between the lower and the upper tail dependence.[14] In contrast, Archimedean copulas allow for asymmetric dependence structures. Prominent representatives are the Gumbel, Clayton and Frank copulas.[15]

For checking the adequacy of specific copula assumptions, we again use a goodness-of-fit test. We employ an approach based on the empirical copula. This approach measures the deviation between the empirical copula and the supposed copula. The null hypothesis

[13] See McNeil, Frey, and Embrechts (2005, p. 275).
[14] The normal copula does not exhibit tail dependence.
[15] A detailed introduction to copula functions is given, for example, in McNeil et al. (2005) and Nelson (2006).

contains the supposed copula $H_0 : C \in \mathcal{C}_0$, which is compared with the empirical copula

$$C_T(\mathbf{u}) = \frac{1}{T}\sum_{t=1}^{T} 1(\hat{U}_{t,1} \leq u_1, ..., \hat{U}_{t,d} \leq u_d) \text{ with } \mathbf{u} = (u_1, ..., u_d) \in [0,1]^d \tag{2.14}$$

where $\hat{\mathbf{U}}_t = (\hat{U}_{t,1}, ..., \hat{U}_{t,d}) = \frac{\hat{\mathbf{R}}_\mathbf{t}}{T+1}$ are the empirical pseudo observations and $\hat{\mathbf{R}}_t$ denotes the vector of ranks of all components at time t. The empirical copula is compared with the estimated copula $C_{\hat{\theta}_T}$ under the null hypothesis. For estimating the parameter vector $\hat{\theta}_T$ of the supposed copula, a variety of methods exists. We use the canonical maximum likelihood estimation (also called maximum pseudo-likelihood).[16] For this method, there is no need to specify the parametric form of the marginal distributions because these are replaced by the empirical marginal distributions. Thus, only the parameters of the copula function have to be estimated by maximum pseudo-likelihood (see Cherubini, Luciano, and Vecchiato (2004, p. 160)). The employed goodness-of-fit test based on the empirical copula uses the Cramér/von Mises[17] test statistic, which is given by

$$S_T = T\int_{[0,1]^d} \left(C_T(\mathbf{u}) - C_{\hat{\theta}_T}(\mathbf{u})\right)^2 dC_T. \tag{2.15}$$

High values of S_T correspond to a large distance between the empirical and the supposed copula and, hence, lead to a rejection of the null hypothesis. In simulation-based power comparison studies, this method delivers more reliable results than many other goodness-of-fit test procedures (see, for example, Berg (2009) and Genest et al. (2009)). In the case that we cannot reject all but one copula functions, we apply the information criteria Akaike Information Criterion (AIC) and Bayesian Information Criterion (BIC) in order to find the best compromise between good approximation and compact dimensioning. The AIC is given by

$$\text{AIC} = -2 \cdot l_C + 2 \cdot k_C \tag{2.16}$$

where l_C stands for the log-likelihood function of the fitted copula C and k_C describes the number of estimated parameters in copula C. Due to the fact that the AIC tends to overparameterize the model,[18] the BIC

$$\text{BIC} = -2 \cdot l_C + k_C \cdot \ln\{T\} \tag{2.17}$$

can be applied, where the added parameter T represents the sample size.

2.4 Default and migrations thresholds

For simulating the obligors' credit quality at the end of the risk horizon in a CreditMetrics-style model, we need the asset return thresholds that correspond to rating migrations. In contrast to the original CreditMetrics credit portfolio model, we cannot just assume that the asset returns are standard normally distributed and compute the asset return

[16] We apply the function gofCopula of the package copula in R in order to estimate the copula parameters as well as to perform the goodness-of-fit test.

[17] See Genest, Rémillard, and Beaudoin (2009, p. 201).

[18] See Hill, Griffiths, and Lim (2011, p. 238).

thresholds by means of the inverse cumulative density function of the standard normal distribution and the rating-specific unconditional default and migration probabilities (see Table 1 in Section 3). The reason for this is that, below, a reverse stress test scenario will be defined by a combination of realizations of the systematic risk factors that cause a specified loss. Based on the multivariate probability distribution of these systematic risk factors, the most likely reverse stress test scenario is computed out of the set of all reverse stress test scenarios. As these systematic risk factors influence the obligors' credit qualities (see (2.8)) and, hence, the value of the bank's portfolio, for computing the asset return thresholds, we have to use the simulated empirical inverse marginal distribution function of the obligors' asset returns that results from the multivariate probability distribution of the systematic risk factors.

3 Data

The objective of this section is to describe the used data. In general, it is desirable to use as many data points as possible. Unfortunately, two problems can arise: First, the economic relation may change through time and, hence, old data may be no longer appropriate for the assumed model. Second, some data points may not be available in lower frequencies. This leads to the unfavorable situation that all data has to be used in the lowest available frequency. In our case, the estimation of the risk factor sensitivities in (2.8) requires the exact number of obligors and their number of defaults for a given time period (see (2.11)). This data is only available on an annual basis. Hence, we are restricted to using exclusively annual data.[19] Moreover, the used time series of risk-free interest rates are available from 1983 and default data from 1981. Thus, we have to cut down the time series by data beginning from 1983.

First, we use the yearly log-returns of the U.S. GDP as the economic indicator $X(t)$. Second, we employ the yearly log-returns of the S&P 500 index which are expected to be more volatile on changes in market conditions than the log-returns of the U.S. GDP (see Figure 3). Both time series are obtained from Datastream.[20]

[19] Of course, we could calculate some intermediate steps with unnecessary precision by using data of a higher frequency (e.g., estimating the principal components on a daily basis), but this would not increase the number of available data points in the linear factor model and would require a subsequent adjustment to annual data. By simulating additional data points (e.g. via bootstrapping), we would substitute one kind of estimation risk (uncertainty in model parameters) for another (simulation uncertainty).

[20] The internal codes in Datastream are USGDP...D and S&PCOMP.

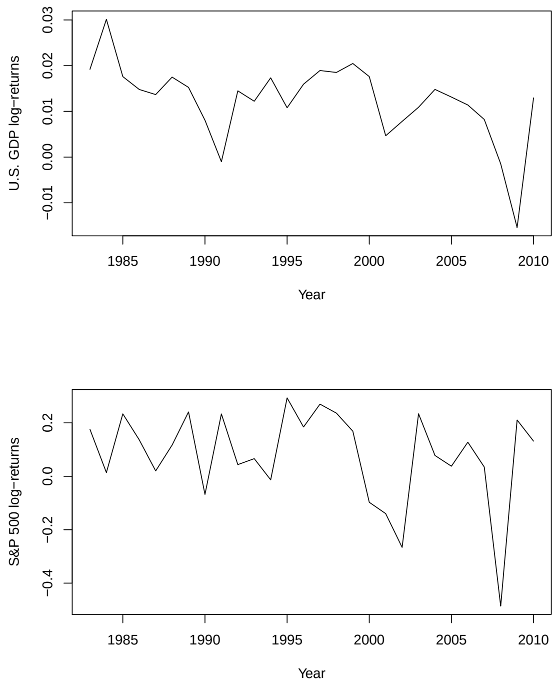

Figure 3: Annual U.S. GDP log-returns and S&P 500 log-returns from 1983 to 2010 obtained from Datastream.

For estimating the principal components, we use annually obtained end-of-year yields of U.S. Treasury Bills (3M, 6M, 1Y) and of U.S. Treasury Bonds (2Y, 3Y, 5Y, 7Y, 10Y, 30Y) ranging from 1983 to 2010 which were provided by Datastream.[21] To ensure stationarity, we calculate percentage changes[22]

$$\Delta r_q(t) = \frac{r_q(t) - r_q(t-1)}{r_q(t-1)}, t \in \{2, ..., T\} \quad \forall \text{ times to maturity } q. \tag{3.1}$$

If necessary, linear interpolation is used to compute the risk-free interest rates for various times to maturity that are needed for discounting in (2.1) and (2.3).

For estimating the linear factor model, we take default data from the annual default report of Standard & Poor's (2011a). The dataset uses empirical data ranging from 1983 to 2010 and contains companies from all over the world. However, it is very likely that most defaults are caused by U.S. companies.[23] As the historical default rates for higher (less risky) rating grades are low and, in some cases, zero, the data is aggregated into the

[21] The internal codes are FRTCM3M, FRTCM6M, FRTCM1Y, USBDS2Y, USBDS3Y, USBDS5Y, USBDS7Y, USBD10Y and USBD30Y.

[22] Otherwise, the null hypothesis that the time series contain unit roots cannot be rejected at reasonable significance levels by the ADF test.

[23] An earlier report of Standard & Poor's (2003, p. 8) made of breakdown according to various regions and shows that most defaults are caused by U.S. companies. A quite similar dataset from Moody's (2011) shows that 84% of defaults are triggered by North American companies for the period from 1986 to 2010. Furthermore, worldwide and U.S. default rates are highly correlated. For the period 1983 to 2010, we calculated a correlation of 97.41% when using data from Standard & Poor's.

two broad rating categories, Investment Grade and Speculative Grade. These are taken to be representative of the assumed homogeneous initial credit qualities AA and BB, respectively, of the obligors in the stylized bank portfolio. The historical default rates for these two broad rating categories are shown in Figure 4.

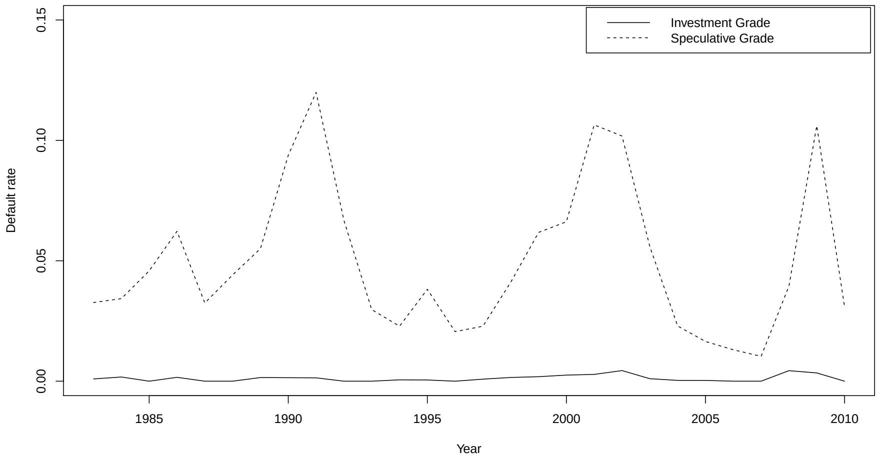

Figure 4: Historical default rates from 1983 to 2010 for Investment Grade and for Speculative Grade obligors. The data was taken from the annual default report of Standard & Poor's (2011a) and considers various sectors from companies all over the world.

The necessary migration and default probabilities (see Section 2.4) over a one-year risk horizon are also provided by Standard & Poor's[24] and summarized in Table 1.

	AAA	AA	A	BBB	BB	B	C-CCC	Default
AAA	90.86%	8.35%	0.56%	0.05%	0.08%	0.03%	0.05%	0.00%
AA	0.59%	90.14%	8.52%	0.55%	0.06%	0.08%	0.02%	0.02%
A	0.04%	1.99%	91.64%	5.64%	0.40%	0.18%	0.02%	0.08%
BBB	0.01%	0.14%	3.96%	90.49%	4.26%	0.71%	0.16%	0.27%
BB	0.02%	0.04%	0.19%	5.79%	83.97%	8.09%	0.84%	1.05%
B	0.00%	0.05%	0.16%	0.26%	6.21%	82.94%	5.06%	5.32%
C-CCC	0.00%	0.00%	0.22%	0.33%	0.97%	15.20%	51.24%	32.03%

Table 1: One-year migration probabilities for the period 1981-2010 taken from Standard & Poor's (2011a) and adjusted for rating withdrawals. The data set contains companies from all over the world, but focuses on U.S. companies.

Furthermore, we have to estimate the average credit spread for each rating category. Credit spread data is provided by Datastream and obtained from straight U.S. corporate bonds which have (as well as our assumed bank portfolio) a time to maturity ranging from 2012 to 2024. The credit spread is calculated as the yield difference of the mid price over a similar sovereign bond.[25] Bonds with a negative credit spread were omitted,[26] half

[24]Data was adjusted for rating withdrawals.

[25]Datastream uses a linear combination of sovereign bonds in order to match the maturities of corporate bonds precisely.

[26]A negative spread can be explained by low liquidity shortly before the maturity date. If a bond is not traded on a day, the last observed price is taken as the current price. Therefore, the bond price does not converge against the face value, and, for bonds priced above their face value, a negative yield (and a negative credit spread) can be calculated.

notches were upgraded (in the case of -) or downgraded (in the case of +). Finally, 2,350 bonds remained. For every rating grade, the credit spread was calculated as the median to ensure an increasing credit spread with worsening rating grade. Table 2 shows the median credit spreads for all rating grades.

Rating	No. of bonds	Credit spread (in bps)
AAA	17	59
AA	57	91
A	639	132
BBB	779	208
BB	348	465
B	355	670
C-CCC	155	959

Table 2: Rating-specific median credit spreads from straight U.S. corporate bonds observed on 17 September 2012 with a time to maturity ranging from 2012 to 2024. The credit spread is calculated as the yield difference of the mid price over a similar sovereign bond. Bonds with a negative credit spread were omitted, half notches were upgraded (in case of -) or downgraded (in case of +).

4 Results

In this section, first, the results for the model calibration are described and, second, the reverse stress test results are presented and discussed.

4.1 Model calibration

In order to determine the number p of principal components to incorporate in the model, we refer to the Kaiser criterion,[27] which recommends using, in the case of a variance-covariance matrix, principal components with an eigenvalue exceeding the mean of the eigenvalues. Following the Kaiser criterion leads to the use of the first two principal components as risk factors for the reverse stress test (instead of all yield-to-maturities with different times to maturity). These explain 96.72% of the total variance; the first three principal components would have explained 99.49%.[28] Figure 5 visualizes the first three principal components for times to maturity ranging from 3 months to 30 years (corresponding to the coefficients $c_{j,q}$ for $j \in \{1, 2, 3\}$ in (2.9)).

[27] See Kaiser (1960).

[28] The third principal component is mentioned and visualized due to the fact that it is used in studies modeling stochastic movements of the term structure of risk-free interest rates by principal components (see, for example, Litterman and Scheinkman (1991), Knez, Litterman, and Scheinkman (1994) and Heidari and Wu (2003)). Nevertheless, for the reverse stress test, we omit it for three reasons: First, the Kaiser criterion proposes the use of only the first two principal components. Second, the maximum likelihood estimation (see (2.11) in conjunction with (2.12)) with an additional risk factor would have been more complex and, third, the evaluation of the risk factor space would have required higher computational effort.

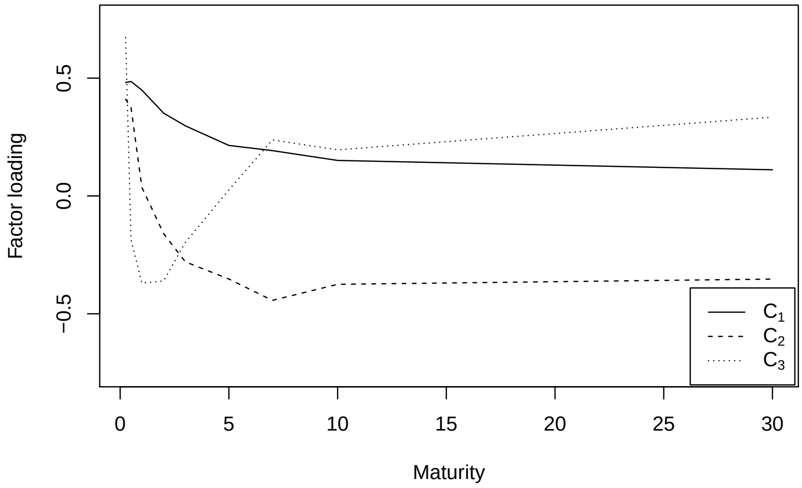

Figure 5: Factor loadings for each maturity of the risk-free interest rates for the first, the second and the third principal components. Principal components $C_j, j \in \{1,2,3\}$, can be interpreted as vectors of factor loadings $c_{j,q}, q \in \{1,2,...,9\}$, for the corresponding interest rates (see (2.9)). For a given (stressed) C_j, the impact on the change of the q-th risk-free interest rate is given by its factor loading $c_{j,q}$ (see (2.10)).

The principal components possess an economic interpretation:[29] The first principal component is a weighted sum of interest rate changes with the same sign for all maturities and can be interpreted as the level of the change in the term structure. The second principal component weights interest rate changes for short maturities with a positive sign and interest rate changes for medium as well as for long maturities with a negative sign and, thus, can be understood as the slope of the interest rate curve. The third principal component associates positive signs with short-term and long-term interest rate changes and associates negative signs with medium-term interest rate changes. Therefore, it can be interpreted as a measure of the curvature.

For the two broad rating categories $i \in \{1,2\} = \{\text{Investment Grade}, \text{ Speculative Grade}\}$, the default barrier $R_{i,K}$ and the vector $(\rho_{i,Z}, \rho_{i,X}, \rho_{i,C_1}, \rho_{i,C_2})$ of asset return sensitivities with respect to the systematic risk factors (see (2.8)) are estimated by maximum likelihood (see (2.11) in conjunction with (2.12)). The time series of realizations of the principal components are calculated from empirical observations of interest rate percentage changes as set out in (2.2). As mentioned above, we use the log-returns of the U.S. GDP and the log-returns of the S&P 500, respectively, within the period $[t, t+1)$ as the economic indicator $X(t)$ in the linear factor model for the asset returns as given in (2.8).[30] The results

[29] See Litterman and Scheinkman (1991, pp. 57-58).

[30] In focusing on GDP and the interest rates (principal components of the risk-free interest rates) as stressed macroeconomic systematic risk factors, we follow Virolainen (2004) and Sorge and Virolainen (2006). Other studies add additional risk factors like commodity prices (see, for example, Misina, Tessier, and Dey (2006)) or credit spreads (see, for example, Avouyi-Dovi, Bardos, Jardet, Kendaoui, and Moquet (2009)). Of course, many other macroeconomic risk factors might also be relevant to explaining defaults (such as industry production or money supply indicators; see, for example, Dorfleitner et al. (2012)).

of the maximum likelihood estimation for the asset return sensitivities are summarized in Table 3.[31]

		$Z(t)$	$X(t)$	$C_1(t)$	$C_2(t)$
GDP	Investment Grade	0.0383	3.3087	0.1749***	0.2524**
	Speculative Grade	0.0557***	7.8860**	0.0963*	0.1925*
S&P 500	Investment Grade	0.0200	0.6643**	0.1056	0.3217***
	Speculative Grade	0.0583***	0.0881	0.1116**	0.2563**

Table 3: Asset return sensitivities with respect to the systematic risk factors as specified in (2.8) for the rating categories Investment Grade and Speculative Grade using annual U.S. GDP and S&P 500 log-returns as economic indicator $X(t)$. The symbols *,** and *** denote significance at 10%, 5% and 1% level.

For Investment Grade as well as for Speculative Grade, the sign for the economic indicator $X(t)$ is economically resonable. For the relationship between asset returns and interest rates, especially principal components of the term structure of interest rates, it is not obvious which sign would be economically reasonable: On the one hand, increased interest rates lead to more expensive loans and therefore should be negatively related to asset returns and, thus, the obligors' credit qualities. On the other hand, raising (short-term) interest rates by central banks is a tool to slow booming economies down in order to control inflation. This explanation is in line with our estimation result in which an increase of the first and the second principal components leads to higher (short-term) interest rates and is positively related to asset returns. The significance of the variables depends on the model specification. Finding significant variables for Investment Grade obligors proves to be rather difficult and only two risk factors can be stated as statistically significant in each of the two specifications.[32] For Speculative Grade obligors, the situation is different. All risk factors have a significant impact when using the specification with U.S. GDP log-returns, whereas three variables prove to be significant in the S&P 500 specification. However, the known criticism with respect to stress tests that statistical relationships can change in an unpredictable manner in a crisis (see, for example, Alfaro and Drehmann (2009)), also applies to our framework. For example, we cannot exclude the possibility that, in times of stress, the sensitivities of the asset returns with respect to the systematic risk factors change or that the relationship between asset returns and systematic risk factors becomes non-linear. This uncertainty is part of the model risk of quantitative reverse stress tests that the senior bank management has to keep in mind when critically

However, any additional macroeconomic risk factor that we add to the linear factor model explaining the obligors' asset returns complicates the reverse stress test due to computational issues. Thus, we face the classical conflict between accuracy and practicability for the desired purpose.

[31]The maximization was performed using the function constrOptim in R and is regarded as numerically stable. Calculations were performed using the Nelder-Mead method (see Nelder and Mead (1965)) with different initial values. Numerical issues due to the improper integral were considered, too. For the integral $\int_{-\infty}^{+\infty}\binom{N_i(t)}{d_i(t)}q_i(z,x(t),c_1(t),c_2(t))^{d_i(t)}(1-q_i(z,x(t),c_1(t),c_2(t)))^{N_i(t)-d_i(t)}\phi(z)dz$, we substituted $y = \Phi(z)$ and $\frac{dy}{dz} = \phi(z)$, respectively. This leads to the expression $\int_0^1\binom{N_i(t)}{d_i(t)}q_i(\Phi^{-1}(y),x(t),c_1(t),c_2(t))^{d_i(t)}(1-q_i(\Phi^{-1}(y),x(t),c_1(t),c_2(t)))^{N_i(t)-d_i(t)}dy$. The following optimization delivered the same result as that one using the improper integral.

[32]A possible explanation is that the parameter's variance is increased due to several observations without defaults. However, it turns out that the coefficients are numerically stable (see Footnote 30) and have the correct sign.

examining the results of a reverse stress test. Furthermore, as in the above discussion with respect to the asset return equations, the multivariate distribution of the risk factors will also typically change in a crisis. Thus, one might think that it is necessary to model multivariate distributions conditional to the extent that the systematic risk factors are stressed during the reverse stress test procedure. However, first, given the available data, the estimation of such a conditional multivariate distribution is not realistic. Second, it is not necessary for the reverse stress test because this kind of model risk influences neither the set of reverse stress scenarios nor the determination of the most likely reverse stress test scenario. The latter point is true because, in determining the probability of occurrence, for the various risk factor combinations we would have to weight the conditional probabilities for the systematic risk factors with the probabilities that a specific degree of a crisis or stress occurs. This is the same as working directly with the unconditional multivariate distribution for the systematic risk factors.[33]

In the next step, we determine the multivariate distribution of the risk factors. First, we test the null hypothesis of normality for the log-returns of the U.S. GDP and the S&P 500 and for the first two principal components by means of the Kolmogorov-Smirnov test and the Jarque-Bera test.[34] The empirical data is visualized in Figure 6 using a QQ plot. While normality seems to be justified in the center of the distribution, the tails differ greatly from this assumption.

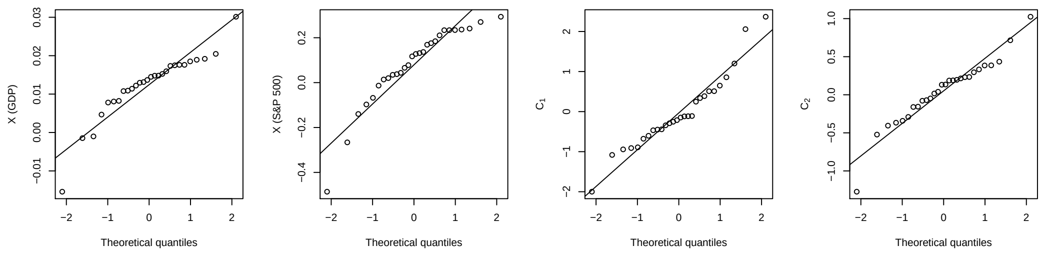

Figure 6: Quantiles of the empirical distribution functions of the systematic risk factors (U.S. GDP log-returns or S&P 500 log-returns and the first and the second principal components) are plotted against quantiles of the normal distribution. Normality seems to be justified in the center of the distributions, the tails seem to differ from this assumption.

As Table 4 shows, the results of the visual inspection are only partly confirmed by the statistical tests. While the Kolmogorov-Smirnov test does not reject the normality assumption, the Jarque-Bera test rejects the null hypothesis for the U.S. GDP log-return, the S&P 500 log-return and the second principal component at the 1% and 5% level, respectively. The Jarque-Bera test calculates skewness and kurtosis of the empirical data and carries them into the test statistic and, hence, quickly rejects normality in the case of supposed fat tails.[35]

[33] Nevertheless, as we model returns as systematic risk factors, of course, the current level of the term structure of risk-free interest rates, to which the stressed principal components are applied (see (2.2)), is considered for discounting the bank's assets and liabilities for the purpose of the reverse stress test.

[34] The latent systematic credit risk factor Z is assumed to be standard normally distributed.

[35] Figure 6 shows that the data for U.S. GDP log-returns includes exactly one outlier (realization in 2009). The same is true for the data of the S&P 500 log-returns (realization in 2008) and the second

	X (GDP)	X (S&P 500)	C_1	C_2
D	0.1719	0.1397	0.1764	0.1132
p-value(D)	0.3400	0.5964	0.3108	0.8266
JB	17.8638***	15.5634***	3.6383	7.3797**
p-value(JB)	$1.3210 \cdot 10^{-4}$	$4.1730 \cdot 10^{-4}$	0.1622	0.0250

Table 4: p-values and test statistics of the Kolmogorov-Smirnov test and the Jarque-Bera test for empirical observations of the risk factors. The symbols *,** and *** denote significance at 10%, 5% and 1% level.

In order to take account of this kind of model risk when carrying out the reverse stress test, we proceed as follows. First, we assume normality of all four systematic risk factors. While the latent systematic credit risk factor Z is assumed to be standard normally distributed, the mean and the variance of the other systematic risk factors are estimated from the data by the method of moments. For the mean, this yields

$$\hat{\boldsymbol{\mu}} = \begin{pmatrix}\hat{\mu}_{X(\text{GDP})} & \hat{\mu}_{X(\text{S\&P 500})} & \hat{\mu}_{C_1} & \hat{\mu}_{C_2}\end{pmatrix} = \begin{pmatrix}0.0124 & 0.0792 & -0.0330 & 0.0512\end{pmatrix} \quad (4.1)$$

and for the variance

$$\hat{\boldsymbol{\sigma}}^{\mathbf{2}} = \begin{pmatrix}\hat{\sigma}^2_{X(\text{GDP})} & \hat{\sigma}^2_{X(\text{S\&P 500})} & \hat{\sigma}^2_{C_1} & \hat{\sigma}^2_{C_2}\end{pmatrix} = \begin{pmatrix}7.1 \cdot 10^{-5} & 0.0304 & 0.8125 & 0.1761\end{pmatrix}. \quad (4.2)$$

Second, we employ extreme value theory to take account of extreme tail events. More precisely, we use methods based on threshold exceedances for the tails of those risk factors for which normality was rejected by the Jarque-Bera test. Tail events are especially important for us since we want to capture extreme scenarios. The Jarque-Bera test rejects normality for the U.S. GDP log-return, the S&P 500 log-return and for the second principal component. To take this into account, we assume the left tail of the distribution of U.S. GDP log-return and of the S&P 500 log-return, respectively, and both tails of the distribution of the second principal component to follow the generalized Pareto distribution (GPD).[36]

The GPD tail and the normally distributed center are connected by the threshold u which is determined by mean excess plots.[37] The threshold u has to be chosen in such a way that the graph of the mean excess function for $u' > u$ is (approximately) linear.[38] Figure 7 shows the mean excess plots for the left tail of the U.S. GDP log-return, for the left tail of the S&P 500 log-return and for the left as well as right tails of the second principal component.

principal component (realization in 2009) that also includes one outlier. When omitting these outliers, we could not reject normality for all risk factors at reasonable significance levels.

[36]Modeling the right tail of the distribution of the U.S. GDP log-return and the S&P 500 log-return, respectively, is not necessary because we are interested in scenarios generating a sufficiently large loss. Thus, due to the positive sign of the asset return sensitivity with respect to the U.S. GDP log-return and the S&P 500 log-return, large U.S. GDP or S&P 500 log-return increases are less relevant. The second principal component, in contrast, has an ambiguous effect on losses because it weights interest rate changes with a short time to maturity with a positive sign and interest rate changes with a long time to maturity with a negative sign. The net effect depends on the portfolio sensitivities towards interest rates for different times to maturity and, therefore, both tails should be modeled by the GPD.

[37]These are graphs that map for every u a mean excess function $\mathbb{E}[X - u | X > u]$ (see, for example, Ghosha and Resnick (2010)). For an application, see, for example, Gourier, Farkas, and Abbate (2009).

[38]This is required due to the linearity of the mean excess function of the GPD.

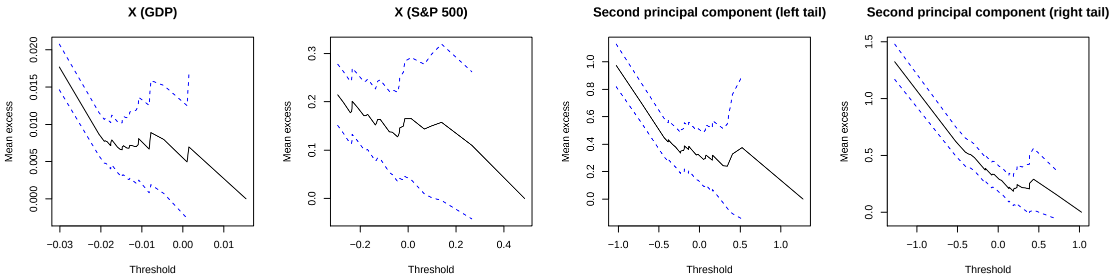

Figure 7: Mean excess plots for the left tail of the U.S. GDP log-return (first), for the left tail of the S&P 500 log-return (second) and the left as well as the right tail of the second principal component (third, fourth). The dashed line indicates the 95% confidence level. Data for U.S. GDP, S&P 500 and the left tail of the second principal component was transformed by multiplying by -1.

In Figure 7, it is notable that a threshold in the interval $[-0.005, 0.005]$ for the U.S. GDP log-return and a threshold of around $[-0.25, -0.1]$ for the S&P 500 log-return are reasonable choices.[39] For the tails of the second principal component, the excess return function seems to be linear when reaching the interval $[0.4, 0.6]$ ($[-0.6, -0.4]$, respectively). As the dataset consists of only 28 observations, we have to choose the thresholds in such a way that, first, the mean excess functions become linear, and, second, the estimation yields plausible results for the parameters of the GPD. For this, the estimation should be based on at least three observations.[40] These considerations let us choose the threshold $u = 0.00$ (U.S. GDP log-return, left tail), $u = -0.13$ (S&P 500 log-return, left tail), $u^l = -0.35$ (second principal component, left tail) and $u^r = 0.35$ (second principal component, right tail). The resulting cumulative density function $F_2(x)$ for the U.S. GDP log-return is given by[41]

$$F_2(x) = \begin{cases} \Phi(0.00)\big(1 + 0.5703\frac{|x-0.00|}{0.0032}\big)^{-\frac{1}{0.5703}} & , x < 0.00 \\ \Phi(x) & , x \geq 0.00. \end{cases} \tag{4.3}$$

and for the S&P 500 log-return, by

$$F_2(x) = \begin{cases} \Phi(-0.13)\big(1 + 0.2139\frac{|x-(-0.13)|}{0.1315}\big)^{-\frac{1}{0.2139}} & , x < -0.13 \\ \Phi(x) & , x \geq -0.13. \end{cases} \tag{4.4}$$

The resulting cumulative density function $F_4(c_2)$ for the second principal component is

$$F_4(c_2) = \begin{cases} \Phi(-0.35)\big(1 + 0.7779\frac{|c_2-(-0.35)|}{0.1196}\big)^{-\frac{1}{0.7779}} & , c_2 < -0.35 \\ \Phi(c_2) & , -0.35 \leq c_2 \leq 0.35 \\ 1 - (1 - \Phi(0.35))\big(1 + 0.2257\frac{c_2-0.35}{0.1900}\big)^{-\frac{1}{0.1900}} & , c_2 > 0.35. \end{cases} \tag{4.5}$$

[39] The data was transformed by multiplication with -1.

[40] Three observations ensure that we have more observations than parameters to estimate.

[41] The parameters were estimated by maximum likelihood and by the probability-weighted moment method, respectively. To take estimation risk into account, the most conservative estimates were used (those with the highest parameter ξ indicating a fat tail).

Next, we want to make a judgement on the appropriate copula function. Table 5 shows the results of the goodness-of-fit test based on the empirical copula for various copula functions. As the probability distribution of the test statistic S_T under the null hypothesis is unknown, it has to be computed by bootstrapping.[42] For this, we perform 100,000 simulation runs.

	GDP		S&P 500	
	Cramér/von Mises	p-value	Cramér/von Mises	p-value
Normal	0.0432	0.5435	0.0478	0.4599
$t_{2\text{df}}$	0.0659	0.1760	0.0730	0.1094
$t_{3\text{df}}$	0.0610	0.2089	0.0694	0.1195
$t_{4\text{df}}$	0.0578	0.2474	0.0662	0.1410
$t_{5\text{df}}$	0.0555	0.2455	0.0636	0.1673
Gumbel	0.0825**	0.0455	0.0667	0.1311
Clayton	0.0514	0.2908	0.0496	0.3409
Frank	0.0793*	0.0910	0.0842*	0.0810

Table 5: Cramér/von Mises test statistics and p-values of the goodness-of-fit test based on the empirical copula for various copula functions. The probability distribution of the test statistic S_T under the null hypothesis was computed by bootstrapping. For this, 100,000 simulation runs were performed. The symbols *,** and *** denote significance at 10%, 5% and 1% level.

As can be seen, we can reject the Frank copula only at a significance level of 10% for the U.S. GDP log-return and the S&P 500 log-return specification and, in the case of U.S. GDP log-return, the Gumbel copula at a significance level of 5%. It is not possible on the basis of the employed goodness-of-fit test to draw further conclusions about which copula function best describes the multivariate dependence structure, which is not surprising given only 28 data points. That is why we apply information criteria in order to find the appropriate copula function. The results of the AIC and BIC statistics are summarized in Table 6.

		Normal	$t_{2\text{df}}$	$t_{3\text{df}}$	$t_{4\text{df}}$	$t_{5\text{df}}$	Clayton	Gumbel	Frank
GDP	ML	3.0151	5.0808	5.0902	4.8397	4.6077	1.7631	0.1018	0.1762
	AIC	-0.0302	-2.1616	-2.1804	-1.6794	-1.2154	-1.5261	1.7964	1.6476
	BIC	3.9664	3.1672	3.1484	3.6494	4.1135	-0.1939	3.1287	2.9798
S&P 500	ML	0.7611	1.9542	2.1710	2.0222	1.8572	0.3467	< 0.0001	0.1738
	AIC	4.4778	4.0916	3.6581	3.9556	4.2856	1.3066	> -1.9999	1.6522
	BIC	8.4744	9.4204	8.9869	9.2844	9.6144	2.6382	> 3.3320	2.9844

Table 6: Maximum pseudo-likelihood and information criterion values (AIC and BIC) for various copula functions. As an economic indicator $X(t)$ either the U.S. GDP or the S&P 500 log-returns are used.

For the U.S. GDP log-return specification, the t-copula with three degrees of freedom yields the lowest AIC value. The BIC, however, implies choosing the Clayton copula, which requires only one parameter.[43] In the case of the S&P 500 log-return specification, the optimal choice for both AIC and BIC is the Clayton copula. Thus, we also face model

[42]For a detailed description, see Genest and Rémillard (2008).

[43]The Clayton copula benefits of its sparse parametrization and the comparatively good fit, while the elliptical copulas are punished due to their high number of parameters and the other Archimedean copulas possess a much worse fit.

risk on the level of the multivariate dependence between the systematic risk factors. We take this into account by carrying out the reverse stress test for both copula specifications. The t-copula enables us to model lower and upper tail dependence and, hence, assumes an increased dependence in boom and bust cycles. The Clayton copula, in contrast, exhibits only lower tail dependence and is therefore well suited to modeling an increased dependence of joint low tail events in times of crisis. The estimated parameters and their significance for the chosen copulas are shown in Table 7.[44] As can be seen, only one parameter estimate is significant, which illustrates the considerable estimation risk (on top of the model risk) that we face when performing a reverse stress test.

		Estimate	Standard error	p-value
GDP, t-copula	ρ_{X,C_1}	0.4196**	0.2046 (2.0512)	0.0402
	ρ_{X,C_2}	0.0603	0.2440 (0.2473)	0.8047
	ρ_{C_1,C_2}	-0.2509	0.1770 (-1.4182)	0.1561
GDP, Clayton copula	θ	0.3783	0.2316 (1.6334)	0.1024
S&P 500, Clayton copula	θ	0.1302	0.1641 (0.7934)	0.4276

Table 7: Copula parameters and their significance. The copula parameters and their significance were estimated by maximum pseudo-likelihood. The t-statistics are presented in parentheses. The symbols *,** and *** denote significance at 10%, 5% and 1% level.

In the last step of the model calibration procedure, we perform 1,000,000 draws in order to determine the empirical distribution functions of the obligors' asset returns. Afterward, the default and migration thresholds are chosen in such a way that they coincide with the appropriate quantiles of the empirical distribution functions of the obligors' asset returns (corresponding to the default and migration probabilities for initially AA-rated and BB-rated obligors, respectively, that are presented in Table 1). The results are summarized in Table 8.[45]

[44] The copula parameters and their significance were estimated by maximum pseudo-likelihood and by inverting Kendall's Tau (see, for example, McNeil et al. (2005, pp. 228-237)) using the functions gofCopula and fitCopula of the package copula in R. Since the estimators deviated less than the standard error of each other, we employed only the maximum pseudo-likelihood estimators. The use of different estimation techniques takes account of the estimation uncertainty and serves as an internal robustness check.

[45] For the reverse stress test, we use the default thresholds as specified in Table 8 instead of the estimated ones in (2.11) in conjunction with (2.12).

GDP	Thresholds for obligors with initial rating grade AA							
	Default	C-CCC	B	BB	BBB	A	AA	AAA
t-copula, normal	$\leq$ -3.56	(-3.56,-3.36]	(-3.36,-3.03]	(-3.03,-2.90]	(-2.90,-2.42]	(-2.42,-1.30]	(-1.30,2.60]	$>$ 2.60
t-copula, GPD	$\leq$ -7.29	(-7.29,-4.53]	(-4.53,-3.26]	(-3.26,-3.07]	(-3.07,-2.48]	(-2.48,-1.31]	(-1.31,2.60]	$>$ 2.60
Clayton, normal	$\leq$ -3.54	(-3.54,-3.36]	(-3.36,-3.06]	(-3.06,-2.92]	(-2.92,-2.45]	(-2.45,-1.31]	(-1.31,2.62]	$>$ 2.62
Clayton, GPD	$\leq$ -8.18	(-8.18,-5.20]	(-5.20,-3.39]	(-3.39,-3.16]	(-3.16,-2.53]	(-2.53,-1.33]	(-1.33,2.62]	$>$ 2.62
	Thresholds for obligors with initial rating grade BB							
	Default	C-CCC	B	BB	BBB	A	AA	AAA
t-copula, normal	$\leq$ -2.22	(-2.22,-1.99]	(-1.99,-1.19]	(-1.19,1.67]	(1.67,2.93]	(2.93,3.34]	(3.34,3.67]	$>$ 3.67
t-copula, GPD	$\leq$ -2.26	(-2.26,-2.02]	(-2.02,-1.20]	(-1.20,1.67]	(1.67,2.93]	(2.93,3.34]	(3.34,3.67]	$>$ 3.67
Clayton, normal	$\leq$ -2.24	(-2.24,-2.00]	(-2.00,-1.20]	(-1.20,1.67]	(1.67,2.95]	(2.95,3.37]	(3.37,3.70]	$>$ 3.70
Clayton, GPD	$\leq$ -2.29	(-2.29,-2.04]	(-2.04,-1.21]	(-1.21,1.67]	(1.67,2.95]	(2.95,3.37]	(3.37,3.70]	$>$ 3.70
S&P 500	Thresholds for obligors with initial rating grade AA							
	Default	C-CCC	B	BB	BBB	A	AA	AAA
Clayton, normal	$\leq$ -3.56	(-3.56,-3.35]	(-3.35,-3.05]	(-3.05,-2.91]	(-2.91,-2.43]	(-2.43,-1.29]	(-1.29,2.63]	$>$ 2.63
Clayton, GPD	$\leq$ -9.87	(-9.87,-5.85]	(-5.85,-3.49]	(-3.49,-3.22]	(-3.22,-2.54]	(-2.54,-1.32]	(-1.32,2.64]	$>$ 2.64
	Thresholds for obligors with initial rating grade BB							
	Default	C-CCC	B	BB	BBB	A	AA	AAA
Clayton, normal	$\leq$ -2.32	(-2.32-2.09]	(-2.09,-1.29]	(-1.29,1.59]	(1.59,2.85]	(2.85,3.27]	(3.27,3.58]	$>$ 3.58
Clayton, GPD	$\leq$ -2.37	(-2.37,-2.12]	(-2.12,-1.30]	(-1.30,1.59]	(1.59,2.86]	(2.86,3.27]	(3.27,3.59]	$>$ 3.59

Table 8: Default and migration thresholds for initial rating grades AA and BB for normal marginal distributions with/without GPD tails and for various copula functions. The empirical distribution functions of the obligors' asset returns are simulated with 1,000,000 draws. Afterward, the default and migration thresholds are chosen in such a way that they coincide with the appropriate quantiles of the empirical distribution functions of the obligors' asset returns (corresponding to the default and migration probabilities for initially AA- and BB-rated obligors, respectively, that are presented in Table 1).

4.2 Reverse stress test results

As described in Section 4.1, to consider model risk, the reverse stress test is performed for the U.S. GDP log-return specification with a t-copula and a Clayton copula dependence structure, and for the S&P 500 log-return specification with a Clayton copula dependence structure. The risk factors are assumed to be (marginally) normally distributed with and without heavier GPD tails. We evaluate $\pm$ 4 standard deviations around the expected value (see Section 2.1) and, thus, evaluate over 99.99% of the probability space in the case of normally distributed margins and over 98.30% in the case of heavier GPD tails. Together with the two assumed initial credit qualities (AA and BB, respectively), we consider 12 test specifications. For each specification, we have to evaluate $17^4 = 83,521$ four-dimensional grid points and perform for each grid point a Monte-Carlo simulation to compute the conditional value-at-risk.[46] The risk horizon is $H = 1$ year and the confidence level of the value-at-risk is set to 99%. The linear factor model for the asset returns of initially AA- (BB-) rated obligors (see (2.8)) is assumed to be represented by the corresponding linear factor model of the broader rating categories Investment Grade and Speculative Grade, respectively. Since, when the considered scenario set is finite, it is very likely that no scenario exhausts the capital buffer exactly, we widen our search to the interval plus/minus 5% around the capital buffer B. For our sample bank, the equity value in $t = 0$ and, hence, the capital buffer B, amounts to 236.32 (with a corresponding equity-to-asset ratio of 29.05%) in the case of initially AA-rated obligors, and to 51.26 in the case of initially BB-rated obligors (with a corresponding equity-to-asset ratio of 8.16%).

For initially AA-rated obligors, none of the considered scenarios completely exhausts the capital buffer. In the case of initially BB-rated obligors, however, the set $\mathbf{\Omega}^*$ of reverse stress test scenarios (see (2.6)) is non-empty. The most likely reverse stress test scenarios (based on the various specifications of the multivariate distribution) are shown

[46] A finer grid would have increased the computation time considerably.

in Table 9.[47]

Economic indicator	Copula and margins	z	x	c_1	c_2	Probability
GDP	t-copula, normal	-0.5	-0.0086	-2.3278	-1.0172	$1.3363 \cdot 10^{-5}$
	t-copula, GPD	-0.5	-0.0002	-0.4920	-1.2309	$1.5716 \cdot 10^{-5}$
	Clayton copula, normal	-1.0	-0.0044	-2.3278	-1.0172	$2.2230 \cdot 10^{-5}$
	Clayton copula, GPD	-0.5	-0.0002	-1.4099	-1.2309	$2.3887 \cdot 10^{-5}$
S&P 500	Clayton copula, normal	-0.5	-0.0952	-0.4920	-1.4446	$2.0662 \cdot 10^{-6}$
	Clayton copula, GPD	-0.5	-0.0952	-0.0330	-1.4446	$1.1456 \cdot 10^{-5}$

Table 9: Most likely reverse stress test scenarios for an initially BB-rated portfolio based on various model specifications.

The most probable scenario exhausting the capital buffer consists of a negative value of the latent systematic risk factor, a slight downturn of the economy (U.S. GDP) or a medium downturn of the economy (S&P 500), a general decrease in the level of interest rates (first principal component), and an increased steepness of the interest rate curve through relatively decreasing interest rates for short maturities compared to increasing interest rates for long maturities (second principal component). This result is robust with respect to the employed model specification. Thus, the bank's senior management would obtain a clear signal of the circumstances under which the bank would get into trouble. The probabilities of the occurrence of the most likely reverse stress test scenarios shown in Table 9 depend on the step size of the grid search. Therefore, these probabilities can only be used for finding the most likely scenario within the set $\mathbf{\Omega}^*$ of all identified reverse stress test scenarios, but they have no absolute interpretation.

For initially BB-rated obligors, Figure 8 shows all scenarios which exhaust the bank's initial capital buffer in the case of the t-copula with normally distributed margins and U.S. GDP as the economic indicator.[48] The reverse stress test scenarios are merged to be conditional on some adjacent values of the latent systematic risk factor $Z(t)$. For example, the upper left plot in Figure 8 visualizes all reverse stress test scenarios which are conditional on a value of [-4,-2.5] of the latent systematic risk factor $Z(t)$. We can observe that all calculated reverse stress test scenarios have something in common. A negative value of the second principal component always seems to be necessary to exhaust the bank's capital buffer. The other variables, however, can be substituted for each other. For example, while fixing the first two principal components, a higher value of the latent systematic risk factor reduces the set of the values of the economic indicator to lower values. Moreover, the rather broad interval $\pm 5\%$ around the initial capital buffer B, which is reached by the reverse stress test scenarios, is responsible for a wide range of possible realizations of the risk factors.

[47] No risk factor takes its boundary value.

[48] In total, 4,253 scenarios are classified as reverse stress test scenarios in the case of the t-copula with normally distributed margins and U.S. GDP as the economic indicator. For the t-copula with heavier GPD tails, we got 4,254 reverse stress test scenarios. In the case of the Clayton-copula, 4,239 (normal, GDP), 4,274 (GPD, GDP), 3,718 (normal, S&P 500) or 3,606 (GPD, S&P 500) scenarios exhausted the initial capital buffer B. The reverse stress test scenarios for the other specifications are qualitatively similar.

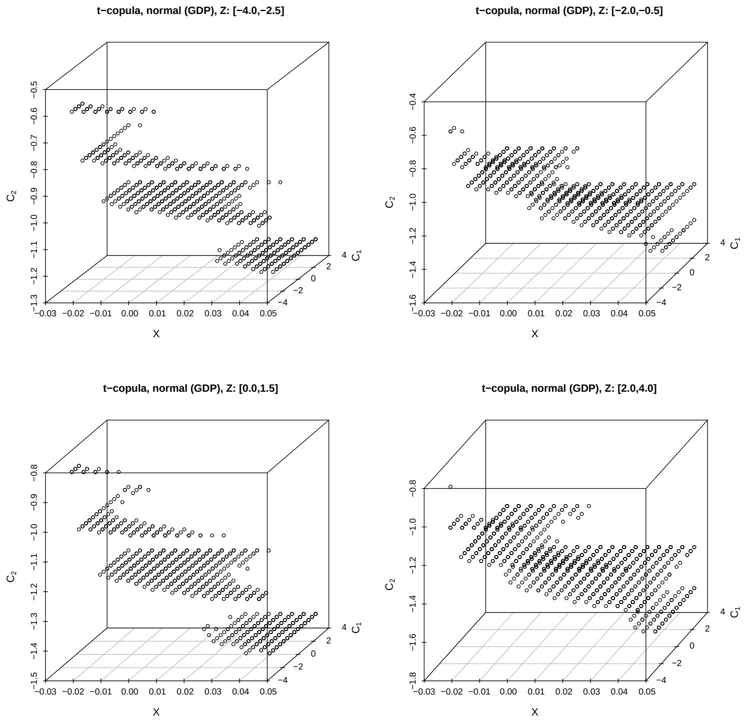

Figure 8: Scenarios which exhaust the bank's initial capital buffer in the case of the t-copula with normally distributed margins and U.S. GDP as the economic indicator (initial rating BB). The reverse stress test scenarios are merged to be conditional on some adjacent values of the latent systematic risk factor $Z(t)$. For example, the upper left plot visualizes all reverse stress test scenarios which are conditional on a value of the latent systematic risk factor $Z(t)$ out of the interval [-4,-2.5].

The stressed term structures of risk-free interest rates in the most likely reverse stress test scenarios are shown in Figure 9. They are calculated as specified in (2.2) from the last observed yield-to-maturities on 4 January 2011, and the realizations of the first and second principal components in the most likely reverse stress test scenarios.

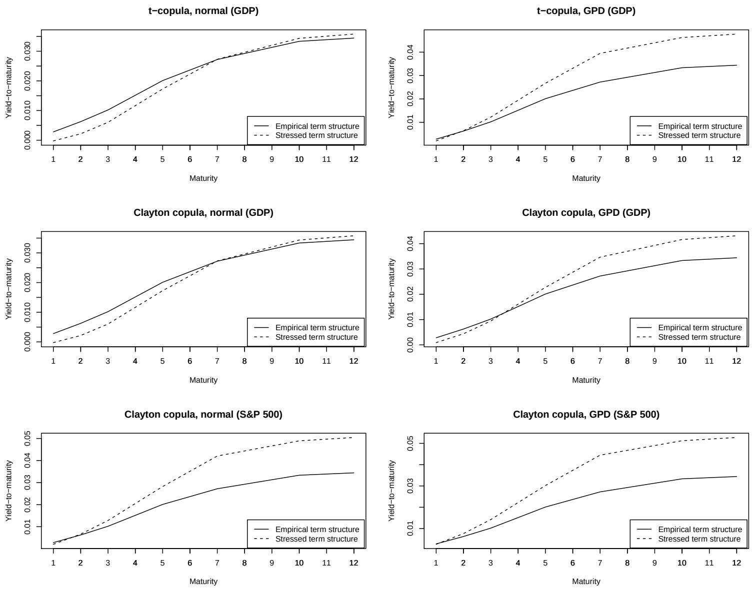

Figure 9: Impact of the most likely reverse stress test scenarios on the term structure of risk-free interest rates. The upper (left: t-copula, normal, right: t-copula, GPD) and middle (left: Clayton copula, normal, right: Clayton copula, GPD) figures show the impact in the case of the U.S. GDP log-return specification. The lower two figures illustrate the case of the S&P 500 log-return specification for a Clayton copula (left: normal, right: GPD). The stressed term structures are calculated as specified in (2.2) from the last observed yield-to-maturities on 4 January 2011, and the realizations of the first and second principal components in the most likely reverse stress test scenarios.

First, changes in the first principal component that correspond to the most likely reverse stress test scenarios push the whole term structure downwards. The impact is rather small because return changes hardly affect the low level of the short-term interest rates and the weights of the long-term interest rates are rather low (see Figure 5). Second, the corresponding changes in the second principal component also lead to decreased short-term interest rates, but to increased long-term interest rates. The sum of these effects leads to decreased short-term interest rates and to considerably increased long-term interest rates. For the equity of the bank, which is performing positive maturity transformation, this induces a double negative impact: Negative cash flows occurring at earlier points in time are discounted by an decreased short-term interest rate, while positive cash flows at later points in time are discounted by considerably increased long-term interest rates. The finding that a steeper term structure of risk-free interest rates represents a reverse stress test scenario for a bank that performs positive maturity transformation stands in contradiction only at first sight to the usual statement that these banks should benefit from a steeper term structure of risk-free interest rates. This statement refers to the improved prospects of future gains when the bank makes new deals (borrowing at short maturities and lending at long maturities). Our definition of reverse stress test scenarios, however, refers to the current composition of the bank's fixed-income portfolio consisting of assets and liabilities and the present value losses of this portfolio caused by shifts in the systematic risk factors.

5 Discussion

Obviously, the data requirements are a bottleneck for the proposed reverse stress test framework. To estimate the linear factor model for the obligors' asset returns (see (2.8)), we used yearly obtained tuples of the number of obligors and the number of defaulted obligors that were provided by Standard & Poor's. In sum, we had only 28 data points, which implies a lot of estimation risk for the risk factor sensitivities in the asset return equations. However, practitioners implementing a quantitative reverse stress test based on internal default data would face a similar problem. Banks can provide internal default data on a quarterly basis,[49] but they also do not have this data for several decades. Even if a bank had quarterly internal default data spanning a time period of, for example, 20 years (corresponding to 80 data points), the question arises as to whether this data is really still representative of the currently used internal rating systems and risk management practices of the bank. Furthermore, using a higher frequency of the default data makes it more probable that there are time periods without any defaults of high quality Investment Grade obligors. This would cause further statistical difficulties. Thus, estimation risk seems unavoidable and has to be taken into account when interpreting the results of the reverse stress test.

For identifying the most likely reverse stress test scenario, we also had to estimate the multivariate distribution of the systematic risk factors.[50] This has also been done based on only 28 data points. These were the realizations of the yearly log-returns of the economic indicator (U.S. GDP and S&P 500, respectively) and the first two principal components of the yearly percentage changes of the risk-free interest rates with different times to maturity. For increasing the number of data points and, thus, enhancing the estimation quality of the multivariate distribution, various strategies exist: 1) Increasing the total time period considered, 2) Using overlapping time windows for computing yearly changes of the systematic risk factors, and 3) Employing risk factor changes over shorter time periods (e.g., quarter or month). Strategy 1) is directly connected with the issue of representativeness of older data. Strategy 2) implies statistical difficulties because the changes of the systematic risk factors are no longer independently distributed. Finally, strategy 3) would yield an estimate of the multivariate distribution of (for example) the quarterly changes of the systematic risk factors. However, the risk horizon used in the internal capital adequacy assessment process that is required by the second pillar of Basel II is typically one year. Thus, based on the multivariate distribution of the quarterly changes of the systematic risk factors, the multivariate distribution of their yearly changes would have to be simulated. This would cause simulation uncertainty. Furthermore, various assumptions would be necessary, for example that the type of the marginal distributions or the correlation parameters of the copula functions do not change with the frequency of the data. In this paper, we have taken into account model and estimation risk with respect to the multivariate distribution of the systematic risk factors by complementing

[49] A higher frequency is not helpful because some of the explanatory variables (e.g., GDP) are available only on a quarterly basis.

[50] Note that we do not need this multivariate distribution for identifying the set $\boldsymbol{\Omega}^*$ of all reverse stress scenarios. Thus, estimation risk with respect to the multivariate distribution of the systematic risk factors does not influence this identification process.

normally distributed margins by GPD tails and by using different types of copula functions with tail dependence.

A further important aspect with regard to quantitative reverse stress tests is their computational tractability. Among other things, this depends on the composition of the portfolio for which the reverse stress test is carried out, on the number of relevant risk factors (which is of course also related to the composition of the portfolio), and on the employed mathematical approach for solving the optimization problem in (2.5). For demonstrating the proposed reverse stress test framework, we assumed a rather simple bank portfolio composition consisting only of zero-coupon bonds. The limited number and choice of the systematic risk factors (a latent systematic risk factor, an economic indicator, and the first two principal components of the term structure of the risk-free interest rates) was made to appear reasonable by this assumption. However, even in this computationally favorable situation, we already reached the limits of our computation capacity.[51] This shows how important an effective reduction of the dimensionality of the space of systematic risk factors is. Furthermore, this shows that, when additional risk factors are unavoidable because of the portfolio composition, the use of more sophisticated techniques for solving the optimization problem (2.5) is essential. For demonstration purposes, we employed a simple grid search, which could be called a computationally intensive, brute-force computation technique.

As already mentioned, the choice of relevant risk factors is crucial in reverse stress testing approaches. On one hand, those risk factors that are most important for the value of the bank's equity have to be captured. On the other hand, the computational burden when solving the inversion problem inherent in reverse stress tests increases exponentially (at least when using simple grid search algorithms) when the number of considered risk factors increases. In our illustrative example, for finding the most important risk factors for explaining profits and losses induced by credit and interest rate risk, we applied a mixture of expert judgement and formal quantitative methods. For explaining the historical default rates, we just assumed (inspired by the corresponding literature on credit risk measurement and stress testing) that a latent risk factor, an economic indicator and yield curve factors are most important. Of course, the selection of the risk factors could also be made exclusively upon a quantitative approach, for example, by employing the stepwise regression (see Rawlings, Pantula, and Dickey (1998, pp. 218-219)) or the Bayesian model averaging (see Sala-i-Martin, Doppelhofer, and Miller (2004)). However, due to the small number of observations, it is very likely that the chosen risk factors would not have been economically plausible and would not have had the best forecast ability. For explaining the direct profits and losses of the zero-coupon bonds induced by yield curve movements, we employ the quantitative method of principal component analysis to reduce the set of all potential risk factors (i.e., the yield-to-maturities of all relevant times to maturity) to a manageable number of risk factors (i.e., the first two principal components).[52] However,

[51] Using an Intel Xeon W3690 processor, 24 GB RAM and Revolution R 5.0.

[52] Effectively, this is also true for explaining the indirect effect on the value of the zero-coupon bonds that yield curve movements have: As interest rate factors drive the obligors' asset returns (see (2.8)), the obligors' ratings (up to a default) and, hence, their credit spreads depend on them. In (2.8), we also employ only the first two principal components instead of the yield-to-maturities of all relevant times to

as Gibson and Pritsker (2001) stress and point out by examples, factors that explain yield curve variations (which are found by principal component analysis) are not necessarily the same factors that explain the variation in the value of a fixed-income portfolio. As an alternative, portfolio-based dimension reduction technique they propose Partial Least Squares.[53] For applying this technique, a sample of portfolio values and the corresponding realizations of the risk factors are needed. Assuming a functional relationship between the portfolio value and the risk factors, the sample of portfolio values can be computed by full portfolio revaluation (given the current portfolio composition). Then, by iterated regressions, Partial Least Squares decompose the risk factor data into orthogonal factors which are ranked in order of importance for the profits and losses of the portfolio (see Gibson and Pritsker (2001)). Unfortunately, it is not obvious how their method can be extended to a situation in which credit and interest rate risk influence the value of a fixed-income portfolio. In our modeling framework, there is no one-to-one relationship between the realizations of the systematic risk factors and the value of the portfolio, but instead, due to the idiosyncratic risk factors, we get a whole probability distribution for the portfolio value given a specific realization of the systematic risk factors.

Our reverse stress test framework for credit and interest rate risk takes a present value perspective, i.e. we analyze how the market value of the bank's equity (defined as the difference between the market value of the bank's assets and the market value of the bank's liabilities) is influenced by changes in the term structure of risk-free interest rates. This is done under the assumption of a static balance sheet, which means that future loans or refinancing operations (with changed interest terms) of the bank are not considered, but only the current composition of the bank's assets and liabilities. For example, this approach is also taken by the banking book interest rate shock analysis that banks have to carry out since Basel II.[54] In practice, however, banks supplement the present value perspective by an earnings-based perspective.[55] In particular, for this approach, banks use the assumption that future loans and refinancing operations are possible according to the changed term structure of risk-free interest rates. It can be shown that under both perspectives, a change in the term structure of risk-free interest rates has the same effect on the bank's equity.[56] The change in the difference of the present values of the bank's assets and liabilities equals the change of the present value of the future net interest income. However, this is only true under the important prerequisite that under the present value perspective, future loans and refinancing operations according to the changed term structure of risk-free interest rates are considered, too. This means that the static balance sheet assumption has to be given up. If this is not the case, both perspectives will usually indicate opposite effects of a change of the term structure of risk-free interest rates. For example, when the term structure of risk-free interest rates gets steeper, this is a

maturity.

[53] A further portfolio-based dimension reduction technique is proposed by Skoglund and Chen (2009). Their approach is based on the Kullback-Leibler divergence.

[54] See BIS (2006a, pp. 212-213).

[55] The Basel Committee on Banking Supervision (BCBS) suggests to consider both perspectives (see BIS (2006b, p. 14)). The European Banking Authority's (EBA) consultation paper on revised guidelines on technical aspects of the management of interest rate risk arising from non trading activities (IRRBB) even proposes using mandatory both methods (see EBA (2013, p. 5)).

[56] See Schmidt (1981).

harmful event for a bank performing positive maturity transformation based on a present value perspective and a static balance sheet assumption (see the discussion at the end of Section 4.2), but it is a positive event within an earnings-based perspective, because the net interest income of new business increases. Thus, as both perspectives have their pros and cons, banks need to run very different stress scenarios for measuring their risks. The most likely reverse stress test scenario found within our proposed framework is a good candidate from a present value perspective, but it should be supplemented by scenarios that are particularly tailored for the earnings-based perspective.

6 Conclusion

In this paper, we have presented a quantitative reverse stress test framework and showed how to implement it empirically. The proposed framework allows us to model interactions between different risk types at the level of the individual financial instruments and risk factors. It is ideally suited solving the inversion problem for computing the set $\mathbf{\Omega}^*$ of all reverse stress test scenarios (see (2.5)) and determining the probability of occurrence of these reverse stress test scenarios.

For a positive maturity-transforming bank, we have determined reverse stress test scenarios for several model specifications, in particular for different marginal distributions for the systematic risk factors and for different multivariate dependence structures. In the case of initially AA-rated obligors, we were unable to detect any reverse stress test scenario at all, but for initially BB-rated obligors, we found that a negative realization of the latent systematic credit risk factor, a slight to medium downturn of the economy as well as decreased risk-free interest rates for short-term maturities and increased risk-free interest rates for long-term maturities represent the most probable reverse stress test scenario. However, the implementation procedure also shows that reverse stress tests are basically exposed to considerable model and estimation risk, which makes numerous robustness checks necessary.

Quantitative reverse stress tests confront banks with considerable challenges. Besides the problem of finding those scenarios in which the viability of the bank is threatened, probabilities of occurrence (or other plausibility criteria) are needed to find the most likely of these scenarios. Further research could deal, for example, with optimization algorithms for finding reverse stress test scenarios that are more intelligent than the simple grid search employed in this paper. This would make it possible to handle extensions with more systematic risk factors and permit the use of a smaller step size. Of course, as long as no reverse stress test standard models are approved, further research using other frameworks is also needed in order to develop appropriate models meeting the regulatory requirements. In particular, frameworks are required that allow a simultaneous reverse stress test for banking book and trading book investments. Ideally, such a framework should also consider contagion effects, systematic recovery risks and changing risk model parameterizations in times of stress. However, it also has to be acknowledged that there are risks (e.g., reputation risk) that can scarcely be evaluated in a quantitative way and, hence, cannot be integrated into a quantitative reverse stress test. This fact makes a combination of quantitative and qualitative reverse stress tests necessary.

A Appendix

A.1 Symbol directory for Section 2

Symbol	Description
B	Bank's capital buffer
B^d	Value of a zero-coupon bond on the asset side
B^l	Value of a zero-coupon bond on the liability side
c_j, c_j^+, c_j^-	Realisation of the j-th principal component
$c_{T_n,j}$	Coefficient of the j-th principal component for the maturity T_n
$C_j, C_j(H)$	j-th principal component of the risk-free interest rate curve (at H)
$\mathbb{E}$	Expected value
H	Risk horizon
$j \in \{1, ..., p\}$	Variable for considered principal components of the interest rate curve
$n \in \{1, ..., N\}$	Variable for positions (obligors) on the asset side
N	Number of positions (obligors) on the asset side
p	Number of considered principal components of the interest rate curve
P	Probability
$q_{1-\alpha}$	Quantile function at the $(1-\alpha)$-level
r_{T_n}	Risk-free interest rate for the maturity T_n
$R(..., T_n)$	Stochastic risk-free interest rate for the maturity T_n
S	Number of Monte-Carlo simulation runs
$S_{\zeta_H^n}, S_{\zeta_H^{bank}}$	Credit spread at H (for asset side position (obligor) n/bank)
t	Time index
T_n	Time to maturity of asset side position (obligor) n
T_v	Time to maturity of liability side position v
$v \in \{1, ..., V\}$	Variable for positions on the liability side
V	Number of positions on the liability side
$V_E(0), V_E(H)$	Market value of bank's equity (initial/at H)
$V_E(H)\|\boldsymbol{\omega}$	Market value of bank's equity at H conditional on $\boldsymbol{\omega}$
$VaR_{\alpha,H}$	Value-at-risk for the α-quantile at H
x, x^+, x^-	Realisation of the economic indicator
X	Economic indicator
z, z^+, z^-	Realisation of the latent systematic risk factor
Z	Latent systematic risk factor
α	Confidence level
δ_n	Recovery rate of the bond issued by obligor n
$\Delta r_{T_n}(H)$	Change in the risk-free interest rate for the time to maturity T_n at H
ζ_0^n, ζ_H^n	Classifier for the rating of asset side position (obligor) n (initial/at H)
μ_{B^d}	Mean of a bond's recovery rate
σ_{B^d}	Standard deviation of a bond's recovery rate
$\boldsymbol{\omega}$	Scenario
$\boldsymbol{\Omega}$	Set of all scenarios
$\boldsymbol{\Omega}^*$	Set of reverse stress test scenarios

Table 10: Overview of the symbols in Section 2.1.

Symbol	Description
$c_j(t)$	Realisation of the j-th principal component (at t)
$c_{q,j}, c_{j,q}$	q/j-th coefficient of the j/q-th principal component
$C_j, C_j(t)$	j-th principal component of the risk-free interest rate curve (at t)
$d_i(t)$	Number of defaulted obligors in rating category i within $[t, t+1)$
$i \in \{1, ..., K\}$	Variable for the rating category
$j \in \{1, ..., m/p\}$	Variable for principal components of the interest rate curve (all/considered)
l_i	Log-likelihood function for rating category i
m	Number of all principal components of the risk-free interest rate curve
$n \in \{1, ..., N\}$	Variable for positions (obligors) on the asset side
N	Number of positions (obligors) on the asset side
$N_i(t)$	Number of obligors in rating category i at t
p	Number of considered principal components of the interest rate curve
P	Probability
$q \in \{1, ..., m\}$	Variable for the q-th interest rate
$q_i(.)$	Conditional default probability for rating category i
r_q	Yield-to-maturity of the q-th interest rate
$R_{n,i}(t)$	Asset return of obligor n in rating category i for the period $[t, t+1)$
$R_{i,K}$	Default barrier for rating category i
t	Time index
t_q	Time to maturity of the q-th interest rate
T	Number of empirical observations
$v \in \{1, ..., V\}$	Variable for positions on the liability side
V	Number of positions on the liability side
$x(t)$	Realisation of the economic indicator at t
$X(t)$	Economic indicator at t
z	Realisation of the latent systematic risk factor
$Z(t)$	Latent systematic risk factor at t
Δr_q	Change in the yield-to-maturity for the time to maturity of the q-th interest rate
$\epsilon_n(t)$	Idiosyncratic risk of obligor n at t
ρ_{i,C_j}	Sensitivity of obligors in rating category i towards C_j
$\rho_{i,X}$	Sensitivity of obligors in rating category i towards X
$\sqrt{\rho_{i,Z}}$	Sensitivity of obligors in rating category i towards Z
$\sqrt{1-\rho_{i,Z}}$	Sensitivity of obligors in rating category i towards idiosyncratic risk
$\phi(.)$	Density function of the normal distribution
$\Phi(.)$	Cumulative density function of the normal distribution

Table 11: Overview of the symbols in Section 2.2.

Symbol	Description
AIC	Akaike Information Criterion
BIC	Bayesian Information Criterion
$\mathcal{C}_0$	Set of copulas
C	Copula
$C_j, C_j(t)$	j-th principal component of the risk-free interest rate curve (at t)
$C_T, C_T(\boldsymbol{u})$	Empirical copula (of $\boldsymbol{u}$)
$C_{\hat{\theta}_T}$	Estimated copula
d	Number of risk factors in a copula function
$G_{\xi,\beta}(y)$	Distribution function of the generalized Pareto distribution
$j \in \{1, ..., p\}$	Variable for considered principal components of the interest rate curve
k_C	Number of parameters of copula C
l_C	Log-likelihood function of copula C
$P(X - u \leq y \vert X > u)$	Conditional probability
$\hat{\boldsymbol{R}}_{\boldsymbol{t}}$	Vector of ranks of the empirical distribution for the risk factors at t
S_T	Cramér/von Mises test statistic
t	Time index
T	Number of empirical observations
$\boldsymbol{u} = (u_1, ..., u_d)$	Empirical distribution function of d risk factors
$\hat{\mathbf{U}}_t = (\hat{U}_{t,1}, ..., \hat{U}_{t,d})$	Empirical pseudo observations of d risk factors at t
$X(t)$	Economic indicator at t
$Z(t)$	Latent systematic risk factor at t
β	Shape parameter of the generalized Pareto distribution
$\hat{\theta}_T$	Parameter vector for the estimated copula
ξ	Scale parameter of the generalized Pareto distribution

Table 12: Overview of the symbols in Section 2.3.

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